

Finite dimensional vertex

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Abstract

The spectrum of linearized excitations of the Type IIB SUGRA on $AdS_5 \times S^5$ contains both unitary and non-unitary representations. Among the non-unitary, some are finite-dimensional. We explicitly construct the pure spinor vertex operators for a family of such finite-dimensional representations. The construction can also be applied to infinite-dimensional representations, including unitary, although it becomes in this case somewhat less explicit.

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1 Introduction

The maximally supersymmetric 10-dimensional background $AdS_5 \times S^5$ [2] of the Type IIB superstring is of the crucial importance in modern string theory, especially in the context of AdS/CFT correspondence. For various reasons, it is important to understand infinitesimal deformations of this background. They correspond to small fluctuations of the classical supergravity fields around their “vacuum” values in $AdS_5 \times S^5$. First of all, one can study them in the linearized approximation, to the first order in the small parameter describing the deviation of the solution from $AdS_5 \times S^5$. We can call such solutions “linearized excitations of $AdS_5 \times S^5$ ”. These linearized excitations can be normalizable and non-normalizable. The normalizable ones correspond to states of the $\mathcal{N} = 4$ SYM on $\mathbf{R} \times S^3$, and the non-normalizable to deformations of the $\mathcal{N} = 4$ SYM [3]. Notice that the symmetry group of $AdS_5 \times S^5$ naturally acts on the space of linearized excitations. It is natural to ask the following question:

- in which representations of the symmetry group of $AdS_5 \times S^5$ do these linearized excitations transform?

For the normalizable excitations, the answer is well-known; it is equivalent to the classification of the local half-BPS operators in $\mathcal{N} = 4$ SYM. The

research program to classify them was initiated in [4]. The answer is a series of unitary representations parametrized by a positive integer. Mathematically, the $\mathcal{N} = 4$ superspace is a super-Grassmannian of embeddings:

$$\mathbf{C}^{2|2} \subset \mathbf{C}^{2+2|4} \quad (1)$$

and the half-BPS operators are holomorphic sections of the n -th power of the Berezinian line bundle. See [5] and references therein.

For the non-normalizable excitations, the situation is more complicated. The space of non-normalizable excitations is not an irreducible representation of the superconformal group, not even of the conformal group. There are subspaces which do not have an invariant complement. To the best of our knowledge, the representation content of the non-normalizable excitations has not been worked out.

By AdS/CFT the non-normalizable excitations correspond to the deformations of the SYM action [3]:

$$S_{YM} \rightarrow S_{YM} + \varepsilon \int d^4x \, \rho(x) \, \mathcal{O}_\Delta(x) \quad (2)$$

where \mathcal{O} is a local operator of conformal dimension Δ and $\rho(x)$ is a density of the conformal weight $\Delta - 4$. Interestingly, for any integer $\Delta \geq 4$ the space of densities has a *finite-dimensional subspace* invariant under the conformal group $SO(2, 4)$. Acting on this space by the supersymmetries we generate a finite-dimensional representation of the full superconformal group $PSU(2, 2|4)$.

In this paper we will construct the pure spinor vertex operators corresponding to some of these finite-dimensional spaces. There are two main motivations. First of all, given the importance of the AdS background in string theory, we would like to know the complete spectrum of the linearized SUGRA on this background, not just unitary representations. Second, from the point of view of the pure spinor formalism, the classification of the finite-dimensional vertices is equivalent to a problem in linear algebra, which is interesting in itself. This may be also related to the $\mathcal{N} = 4$ integrability program along the lines of [6].

There were two previously known examples of a finite-dimensional vertex: the zero mode of the dilaton constructed in [7], and the vertex for the beta-deformation considered in [8]. Here we construct an infinite series of new examples.

Plan of the paper We will construct the universal vertex in the sense of [6] for a specific finite-dimensional representation. We describe this representation in Section 2. Then in Section 3 we discuss (as a conjecture) the SYM interpretation. The construction itself is described in Sections 4, 5, 6 and 7. A partial analysis of the corresponding SUGRA solutions is presented in Section 8. A possible generalization to infinite-dimensional (including unitary) representation is discussed in Section 9. In Section 10 we discuss some representation-theoretic properties of our construction. Open questions are listed in Section 11.

2 Algebraic preliminaries

2.1 Ansatz for the vertex

We will use the following ansatz for the vertex transforming in the representation \mathcal{H} of \mathfrak{g} . For every $\Psi \in \mathcal{H}$ the corresponding vertex is [6]:

$$V[\Psi](g, \lambda) = \langle v(\lambda), g\Psi \rangle \quad (3)$$

where $v(\lambda) \in \mathcal{H}'$ — a constant (*i.e.* independent on g) vector in \mathcal{H}' . Here we denote \mathcal{H}' the space dual to \mathcal{H} :

$$\mathcal{H}' = \text{Hom}_{\mathbb{C}}(\mathcal{H}, \mathbb{C}) \quad (4)$$

2.2 Definition of \mathcal{H}

We will construct $v(\lambda)$ for a specific series of finite-dimensional representations \mathcal{H} , which we will now define.

2.2.1 Parabolic induction

Block structure of $sl(4|4)$ The even subalgebra of $sl(4|4)$ is a direct sum of two Lie algebras:

$$\mathfrak{g}_{\text{even}} = \mathfrak{g}_{\text{up}} \oplus \mathfrak{g}_{\text{dn}} \quad (5)$$

(The real form would be $\mathfrak{g}_{\text{up}} = \mathfrak{u}(2, 2)$ and $\mathfrak{g}_{\text{dn}} = \mathfrak{u}(4)$.) Schematically, in the 4×4 -block notations:

$$\mathfrak{g} = \begin{bmatrix} \mathfrak{g}_{\text{up}} & \mathfrak{n}_+ \\ \mathfrak{n}_- & \mathfrak{g}_{\text{dn}} \end{bmatrix} \quad (6)$$

The \mathbf{n}_- in the upper right corner and \mathbf{n}_+ in the lower left corner are both odd abelian subalgebras $\mathbf{C}^{0|16}$.

Parabolic subalgebra \mathbf{p} Let us denote \mathbf{p} the following parabolic subalgebra of \mathbf{g} :

$$\mathbf{p} = \begin{bmatrix} \mathbf{g}_{\text{up}} & 0 \\ \mathbf{n}_- & \mathbf{g}_{\text{dn}} \end{bmatrix} \quad (7)$$

We will denote:

$$F_{\text{up}} : \text{the fundamental of } \mathbf{g}_{\text{up}} \quad (8)$$

$$F_{\text{dn}} : \text{the fundamental of } \mathbf{g}_{\text{dn}} \quad (9)$$

and $F'_{\text{up}}, F'_{\text{dn}}$ will denote the corresponding dual representations (a.k.a. “antifundamental representations”).

2.2.2 Construction of \mathcal{H} as an induced representation

We will construct a series of representations of \mathbf{g} from a series of finite-dimensional representations of the bosonic subalgebra $\mathbf{g}_{\text{up}} \oplus \mathbf{g}_{\text{dn}}$, using the parabolic induction.

A series of representations of the bosonic algebra \mathbf{g}_{even} Let us consider the following finite dimensional representation of $\mathbf{g}_{\text{up}} \oplus \mathbf{g}_{\text{dn}}$ parametrized by an integer n :

$$L = Y \left(F_{\text{dn}}^{\otimes 2(n+1)} \right) \otimes Y \left((F'_{\text{up}})^{\otimes 2(n+1)} \right) \quad (10)$$

where Y is some specific Young symmetrizer, which acts as follows. The space $F_{\text{dn}}^{\otimes 2(n+1)}$ consists of the tensors $f^{a_1 \dots a_{2(n+1)}}$; the operation Y first antisymmetrizes $[a_1 a_2], \dots, [a_{2n+1} a_{2n+2}]$, and then symmetrizes $(a_1 a_3 a_5 \dots a_{2n+1})$ and $(a_2 a_4 a_6 \dots a_{2n+2})$. Similarly the space $Y(F'_{\text{up}})^{\otimes 2(n+1)}$ is the space of tensors $f_{\alpha_1 \dots \alpha_{2(n+1)}}$ which are antisymmetrized and then symmetrized in the same way.

Induced representation of \mathbf{g} Let us extend L to a representation of the parabolic subalgebra (7) by letting \mathbf{n}_- act as 0.

We will define:

$$\mathcal{H} = \text{Ind}_{\mathbf{p}}^{\mathbf{g}} L \quad (11)$$

where the operation $\text{Ind}_{\mathbf{p}}^{\mathbf{g}}$ is defined as follows:

$$\text{Ind}_{\mathbf{p}}^{\mathbf{g}} L = \mathcal{U}\mathbf{g} \otimes_{\mathbf{p}} L \quad (12)$$

The nilpotent subalgebra \mathbf{n}_- acts trivially on L .

Tensor notations The elements of L are tensors:

$$V_{\alpha_1 \dots \alpha_{2n+2}}^{a_1 \dots a_{2n+2}} \quad (13)$$

with the appropriate symmetry conditions. We will introduce the coordinates $(\theta_+)_a^\alpha$ on \mathbf{n}_+ .

2.3 Dual space \mathcal{H}' and holomorphic vector bundles on G/G_{even}

2.3.1 Duality between induced and coinduced representations

Notice that $v(\lambda)$ in Eq. (3) belongs to the representation dual to \mathcal{H} . The dual representation to $\text{Ind}_{\mathbf{p}}^{\mathbf{g}} L$ is the coinduced representation:

$$\mathcal{H}' = \text{Coind}_{\mathbf{p}}^{\mathbf{g}} L' = \text{Hom}_{\mathbf{p}}(\mathcal{U}\mathbf{g}, L') \quad (14)$$

We consider both $\mathcal{U}\mathbf{g}$ and L' as right \mathbf{p} -modules; \mathcal{H}' can be thought as the space of functions $f : \mathcal{U}\mathbf{g} \rightarrow L'$ satisfying the property:

$$f(\xi\eta) = f(\xi)\eta \text{ for } \xi \in \mathcal{U}\mathbf{g} \text{ and } \eta \in \mathcal{U}\mathbf{p} \quad (15)$$

The duality pairing is:

$$\langle f, \xi \otimes_{\mathcal{U}\mathbf{p}} l \rangle = \langle f(\xi), l \rangle \quad (16)$$

where \langle, \rangle on the RHS is the pairing between L' and L .

Consistency of the definition of pairing Take some $\xi \in \mathcal{U}\mathbf{g}$, $g_{\text{even}} \in \mathcal{U}\mathbf{g}_{\text{even}}$ and $g_- \in \mathcal{U}\mathbf{n}_-$. We get:

$$\langle \Phi_\lambda(\xi g_{\text{even}} g_-), l \rangle = \langle \Phi_\lambda(\xi) g_{\text{even}} g_-, l \rangle = \langle \Phi_\lambda(\xi), g_{\text{even}} l \rangle \quad (17)$$

Invariance under global rotations

$$\langle g \cdot \Phi_\lambda, \xi \otimes_{\mathbf{p}} l \rangle = \langle \Phi_\lambda(g^{-1}\xi), l \rangle = \langle \Phi_\lambda, g^{-1}\xi \otimes_{\mathbf{p}} l \rangle \quad (18)$$

g₀-covariance In particular, consider the case when $g \in \mathcal{U}\mathfrak{g}_{\bar{0}}$ in (18):

$$(h.\Phi_\lambda)(\xi) = \Phi_\lambda(h^{-1}\xi) = \Phi_\lambda(h^{-1}\xi h)h^{-1} \quad (19)$$

For covariance, we want this to be equal to $\Phi_{h\lambda h^{-1}}(\xi)$. Therefore we need to impose the *g₀-covariance condition* on Φ :

$$\Phi_\lambda(\xi) = \Phi_{h^{-1}\lambda h}(h^{-1}\xi h)h^{-1} \quad (20)$$

2.3.2 Relation to [1]

Geometrically \mathcal{H}' can be thought of as the space of holomorphic sections of the vector bundle on the odd Grassmanian G/G_{even} with the fiber L' . Notice that G/G_{even} is the target space of the gauged linear sigma-model of [1]. Although that was our main motivation in using the parabolic induction, the precise relation of our method to the discussion in [1] is not clear to us.

2.3.3 Explicit formulas for the action of global symmetries

Consider $\eta_+ \in \mathfrak{n}_+$ and $\eta_- \in \mathfrak{n}_-$. We get:

$$e^{-\eta_+}\Phi(e^{\theta_+}) = \Phi(e^{\eta_++\theta_+}) \quad (21)$$

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} e^{-t\eta_-}\Phi(e^{\theta_+}) &= \left. \frac{d}{dt} \right|_{t=0} \Phi(e^{t\eta_-}e^{\theta_+}) = \\ &= \frac{1}{2} \left([\theta_+, [\theta_+, \eta_-]] \frac{\partial}{\partial \theta_+} \right) \Phi(e^{\theta_+}) + \Phi(e^{\theta_+}) [\eta_-, \theta_+] \end{aligned} \quad (22)$$

2.4 Coinduced representation is not irreducible

Let us denote F (without lower index) the fundamental representation of the *super*-algebra \mathfrak{g} , and F' the dual (antifundamental) representation. Let us consider the following symmetrized tensor product:

$$\mathcal{T} = Y(F^{\otimes 2(n+1)}) \otimes Y^{\text{tr}}((F')^{\otimes 2(n+1)}) \quad (23)$$

where Y and Y^{tr} are the super-symmetrizers. There is a canonical map:

$$\text{ev} : \mathcal{H} \rightarrow \mathcal{T} \quad (24)$$

It is constructed using the embedding ι :

$$Y((F'_{\text{up}})^{\otimes 2(n+1)}) \otimes Y((F_{\text{dn}})^{\otimes 2(n+1)}) \xrightarrow{\iota} Y(F^{\otimes 2(n+1)}) \otimes Y^{\text{tr}}((F')^{\otimes 2(n+1)}) \quad (25)$$

Using this embedding, ev is defined as the action of the element of $\mathcal{U}\mathfrak{g}$ on the embedded tensor:

$$\text{ev}(\xi \otimes f) = \xi \iota(f) \quad (26)$$

Notice that $\ker(\text{ev}) \subset \mathcal{H}$ is an invariant subspace, but there is no complementary invariant subspace. Therefore, \mathcal{H}' has an invariant subspace consisting of those functionals which vanish on $\ker(\text{ev})$. This subspace consists of the following functionals, using the notations of Section 2.3:

$$f(\xi) = (T\xi)|_{\text{restriction to } L} \quad (27)$$

where $T \in \mathcal{T}'$ — a linear function on \mathcal{T} . On the right hand side we evaluate the action of $\xi \in \mathcal{U}\mathfrak{g}$ on this T , and then restrict the resulting linear functional to $L \subset \mathcal{T}$, where the embedding of L into \mathcal{T} is the ι of (25). We will denote the subspace of functions of the form (27) in the standard way:

$$(\ker(\text{ev}))^\perp = \{f \mid f(\xi) = (T\xi)|_{\text{restriction to } L}\} \quad (28)$$

More explicitly, given a tensor $T_{j_1 j_2 \dots j_{2n}}^{i_1 i_2 \dots i_{2n}}$, we associate to it the following holomorphic section:

$$\Phi[T]_{a_1 \dots a_{2n}}^{\alpha_1 \dots \alpha_{2n}}(\theta_+) = (Te^{\theta_+})_{a_1 \dots a_{2n}}^{\alpha_1 \dots \alpha_{2n}} \quad (29)$$

Such sections form an invariant subspace $(\ker(\text{ev}))^\perp \subset \text{Coind}_{\mathfrak{g}}^{\mathfrak{g}} L'$.

2.5 Properties of $Y(F^{\otimes 2(n+1)})$

Dimension The representation $Y(F^{\otimes 2(n+1)})$ can be identified with the space of traceless symmetric tensors of $so(6)$. The dimension is:

$$\dim Y(F^{\otimes 2(n+1)}) = \frac{(n+2)(n+3)^2(n+4)}{12} \quad (30)$$

For example, when $n = 1$ we get:

$$\begin{array}{c} \square \\ \square \end{array} \otimes \begin{array}{c} \square \\ \square \end{array} = \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} + \begin{array}{cc} \square & \square \\ \square & \square \end{array} + \begin{array}{cc} \square & \square \\ \square & \square \end{array} \quad (31)$$

36 1 15 20

The symmetrization of $(\alpha_1 \alpha_3)$ and $(\alpha_2 \alpha_4)$ removes **15** and **1** and leaves **20**.

Decomposition as a representation of $sp(2) = so(5)$ The tensor product of $n + 1$ vector representations of $so(5)$ has an invariant subspace T_{n+1} , which consists of the symmetric traceless tensors. Its dimension is $\frac{(2n+5)(n+2)(n+3)}{6}$. We have the decomposition:

$$Y(F^{\otimes 2(n+1)}) = \bigoplus_{k=0}^{n+1} T_k \quad (32)$$

$$\frac{(n+2)(n+3)^2(n+4)}{12} = \sum_{k=0}^{n+1} \frac{(2n+5)(n+2)(n+3)}{6} \quad (33)$$

3 Field theory point of view

3.1 A finite-dimensional supermultiplet of deformations

We conjecture that our vertex operator is dual to the finite-dimensional supermultiplet of deformations of the $\mathcal{N} = 4$ SYM generated by the following deformation:

$$S_{YM} \rightarrow S_{YM} + \varepsilon \int d^4x \, \rho(x) \, \text{tr} \, Z^{n+3}(x) \quad (34)$$

where $Z = \Phi^5 + i\Phi^6$ is a complex combination of the SYM scalars and $\rho(x)$ is a density of the conformal weight $n - 1$. Observe that for any integer $n > 0$ the space of densities has a finite-dimensional subspace invariant under the conformal group $SO(2, 4)$. This can be seen for example from the dual AdS picture. In the AdS language, this subspace corresponds to the harmonic polynomials of weight $n - 1$ in \mathbf{R}^{2+4} . This can be also seen as follows. The infinitesimal special conformal transformations act on Z as follows:

$$\delta_K Z(x) = \left((k \cdot x)(x \cdot \partial) - \frac{1}{2}(x \cdot x)(k \cdot \partial) + (k \cdot x) \right) Z(x) \quad (35)$$

Therefore the deformation is invariant with the following transformation rule² for $\rho(x)$:

$$\delta_K \rho(x) = \left((k \cdot x)(x \cdot \partial) - \frac{1}{2}(x \cdot x)(k \cdot \partial) + (1 - n)(k \cdot x) \right) \rho(x) \quad (36)$$

²We could think of $\rho(x)$ as a “spectator field”, or as an x -dependent coupling constant

In other words, we should think of ρ as a “density” of weight $1 - n$. Consider the linear space consisting of the densities $\rho(x)$ of the following form:

$$\rho(x) = \sum_{m=0}^{n-1} q_{n-m-1}((x \cdot x)) p_m(x) \quad (37)$$

where $p_m(x)$ is a homogeneous polynomial in x of weight m , and $q_{n-m}((x \cdot x))$ a polynomial in $(x \cdot x)$ of weight $n - m$ (not necessarily homogeneous; *e.g.* $q((x \cdot x)) = 1$ is OK). One can see this space is closed under the action of the conformal transformations. This means that the (infinite-dimensional) representation of the conformal group $su(2, 2)$ in the space of densities of the weight $1 - n$ for $n > 0$ has an invariant finite-dimensional subspace³ (37).

For example, when $n = 1$ there is a conformally invariant deformation:

$$S_{YM} \rightarrow S_{YM} + \varepsilon \int d^4x \operatorname{tr} Z^4 \quad (38)$$

Let us consider a particular case $n = 1$, which corresponds to $\operatorname{tr} Z^4$.

3.2 Action of R-symmetry on $\operatorname{tr} Z^k$

Let us first consider the action of $so(6)$ on (38). The perturbation transforms in the following representation of $so(6)$:

$$\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \quad (39)$$

which has dimension **105**. The highest weight is:

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline \end{array} \quad (40)$$

Similarly:

$$\operatorname{tr} Z^5 : \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 \\ \hline \end{array} \quad \operatorname{tr} Z^6 : \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ \hline \end{array} \quad \text{etc.} \quad (41)$$

For $\operatorname{tr} Z^4$ there are states constant in AdS. They correspond to:

$$\left(E_{a_1}^{\alpha'} E_{a_2}^{\beta'} E_{a_3}^{\gamma'} E_{a_4}^{\delta'} \right) \otimes \left(Y(e_{\alpha'} \otimes e_{\beta'} \otimes e_{\gamma'} \otimes e_{\delta'}) \otimes \tilde{Y}(e^{b_1} \otimes e^{b_2} \otimes e^{b_3} \otimes e^{b_4}) \right) \quad (42)$$

The state corresponding to $\int d^4x \operatorname{tr} Z^4$ is obtained by taking $(a_1, a_2, a_3, a_4) = (3, 4, 3, 4)$ and $(b_1, b_2, b_3, b_4) = (1, 2, 1, 2)$.

³There is no invariant complementary subspace, therefore the space of densities is not a semisimple representation of $su(2, 2)$.

3.3 Supersymmetric Young diagrams

The diagramm (40) can be embedded in the following supersymmetric Young diagram⁴, using the notations of [10]:

$$\begin{array}{|c|c|} \hline 3 & 3 \\ \hline 4 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \quad (43)$$

(The labels correspond to the highest weight state.) Similarly, we have the following diagrams for $\text{tr } Z^5$, $\text{tr } Z^6$ etc.:

$$\begin{array}{|c|c|} \hline 1_F & 2_F \\ \hline 3 & 3 \\ \hline 4 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1_F & 2_F \\ \hline 1_F & 2_F \\ \hline 3 & 3 \\ \hline 4 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline \end{array} \quad (44)$$

where 1_F means that the index is fermionic, and therefore the highest weight state for $\text{tr } Z^n$ has to have a nontrivial dependence on spacetime coordinates, for $n > 4$.

However, our vertex corresponds to the indecomposable representation (12) (the Kac module) rather than the irreducible representation corresponding to the Young diagram.

3.4 A puzzle

Now let us consider $\text{tr } Z^k$ with $k > 4$. The lowest possible AdS momentum is then $k - 4$. However, it appears that we can construct vertices with lower momentum. For example, consider the following state in the multiplet which should correspond to $\text{tr } Z^5$:

$$\begin{aligned} & \left(E_{a_1}^{\alpha'} E_{a_2}^{\beta'} E_{a_3}^{\gamma'} E_{a_4}^{\delta'} E_{a_5}^{\epsilon'} E_{a_6}^{\zeta'} \right) \otimes \\ & \otimes \left(Y(e_{\alpha'} \otimes e_{\beta'} \otimes e_{\gamma'} \otimes e_{\delta'} \otimes e_{\epsilon'} \otimes e_{\zeta'}) \otimes \right. \\ & \left. \otimes Y^{\text{tr}}(e^{b_1} \otimes e^{b_2} \otimes e^{b_3} \otimes e^{b_4} \otimes e^{b_5} \otimes e^{b_6}) \right) \quad (45) \end{aligned}$$

⁴Notice that the SUSY Young diagrams have been considered previously in the context of AdS/CFT in [9], but in that paper only the products of fundamental representations are needed. Here we discuss Young diagrams involving both fundamental and antifundamental representations.

Here $E_{a_1}^{\alpha'} E_{a_2}^{\beta'} E_{a_3}^{\gamma'} E_{a_4}^{\delta'} E_{a_5}^{\epsilon'} E_{a_6}^{\zeta'}$ is an element of the universal enveloping $\mathcal{U}\mathfrak{g}$. The corresponding vertex is constant in AdS, because all the AdS (greek) indices are contracted. It appears that there is no reason to discard this state. However observe that such states necessarily will have a nonzero contraction of the lower a indices and the upper b indices. It must be true that the vertex for (45) is BRST-exact. But we have not checked it explicitly.

4 BRST operator and parabolic induction

In this and the following sections we will describe the construction of the covariant vertex. We will start by calculating the action of the BRST operator Q in the induced representation.

4.1 The bicomplex $Q_L = Q_{L+} + Q_{L-}$

4.1.1 Anatomy of pure spinor

Consider the “left” pure spinor $\lambda \in \mathfrak{g}_3$. We get:

$$\lambda = \begin{pmatrix} 0 & \lambda_+ \\ \lambda_- & 0 \end{pmatrix} \quad (46)$$

The condition that $\lambda \in \mathfrak{g}_3$ implies:

$$(\lambda_-)_\alpha^a = i \omega^{aa'} (\lambda_+)_{a'}^{\alpha'} \omega_{\alpha'\alpha} \quad (47)$$

where $\omega_{\alpha\beta}$, ω_{ab} is the symplectic form on $\mathbf{C}^{4|4}$ which defines the denominator subalgebra $sp(2) \subset sl(4)$ as explained in [11]. In other words, the choice of ω up to multiplication by a number is equivalent to the choice of a point in $AdS_5 \times S^5$. The purity condition implies that:

$$\lambda_a^\alpha \omega^{ab} \lambda_b^\beta \simeq \omega^{\alpha\beta} \quad (48)$$

$$\lambda_a^\alpha \omega_{\alpha\beta} \lambda_b^\beta \simeq \omega_{ab} \quad (49)$$

In other words, pure spinors parametrize the group manifold of $Sp(2)$. We also observe that (48) \Rightarrow (49), therefore it is enough to impose only 5 pure spinor constraints (see *e.g.* Section 4.2 of [7]).

For brevity we will write $\lambda_a^\alpha = (\lambda_+)_a^\alpha$, *i.e.* drop the subindex $+$. Therefore the BRST operator $Q_L = \lambda^i t_i$ splits into the sum of the two anticommuting nilpotent operators:

$$Q_L = Q_{L+} + Q_{L-} \quad (50)$$

$$Q_{L+} = \lambda_+^i t_i \quad (51)$$

$$Q_{L-} = \lambda_-^i t_i \quad (52)$$

4.1.2 Notations \cap , \cup and $|| \dots ||$

We will denote:

$$X_a \omega^{ab} Y_b = X \cap Y, \quad X^\alpha \omega_{\alpha\beta} Y^\beta = X \cup Y, \quad (53)$$

$$X_{ab} \omega^{ba} = ||X||, \quad Y^{\alpha\beta} \omega_{\beta\alpha} = ||Y|| \quad (54)$$

4.1.3 The structure of Q_{L+}

The action of Q_{L+} on Φ follows from (21):

$$Q_{L+} \Phi_{a_1 \dots a_{2n}}^{\alpha_1 \dots \alpha_{2n}}(\theta_+) = \left(\lambda_a^\alpha \frac{\partial}{\partial (\theta_+)_a^\alpha} \right) \Phi_{a_1 \dots a_{2n}}^{\alpha_1 \dots \alpha_{2n}}(\theta_+) \quad (55)$$

Therefore:

- the action of Q_{L+} coincides with the flat space zero mode BRST cohomology.

We will often abbreviate θ_+ to θ .

4.1.4 The structure of Q_{L-}

The expression for Q_{L-} is somewhat more involved. From (22) we get:

$$\epsilon Q_{L-} \Phi = i \left[\theta \cap \epsilon \lambda \cup \Phi + \Phi \cap \epsilon \lambda \cup \theta - \left((\theta \cap \epsilon \lambda \cup \theta) \frac{\partial}{\partial \theta} \right) \Phi \right] \quad (56)$$

A redefinition of Q and Φ To make the formulas look better, we would prefer to get rid of the i in the RHS of (59). This can be done by the redefinition:

$$\Phi \mapsto \exp \left[\frac{i\pi}{4} (\#\theta) \right] \Phi \quad (57)$$

$$Q \mapsto \exp \left[\frac{i\pi}{4} (\#\theta) \right] Q \quad (58)$$

where $\#\theta$ is the number of θ 's in the expression. After this redefinition, we get:

$$\epsilon Q_{L-} \Phi = \theta \cap \epsilon \lambda \cup \Phi + \Phi \cap \epsilon \lambda \cup \theta - \left((\theta \cap \epsilon \lambda \cup \theta) \frac{\partial}{\partial \theta} \right) \Phi \quad (59)$$

Assuming that Φ is even (*i.e.* contains even number of θ 's) we get:

$$Q_{L-} \Phi = -\theta \cap \lambda \cup \Phi + \Phi \cap \lambda \cup \theta + \left((\theta \cap \lambda \cup \theta) \frac{\partial}{\partial \theta} \right) \Phi \quad (60)$$

4.2 The spectral sequence of $Q_{L+} + Q_{L-}$

Let us look closer at the spectral sequence corresponding to the bicomplex $Q_{L+} + Q_{L-}$. We will consider the filtration by the following degree:

$$\deg(\Phi) = \frac{1}{2} [\text{number of } \theta\text{'s plus number of } \lambda\text{'s in } \Phi] \quad (61)$$

Notice that

$$\deg(Q_{L+} \Phi) = \deg(\Phi) \quad (62)$$

$$\deg(Q_{L-} \Phi) = \deg(\Phi) + 1 \quad (63)$$

The first page of the spectral sequence for $Q_{L+} + Q_{L-}$ is:

$$E_1^{p,q} = \frac{\ker (Q_{L+} : [\lambda^{p+q} \theta^{p-q}] \rightarrow [\lambda^{p+q+1} \theta^{p-q-1}])}{\text{im} (Q_{L+} : [\lambda^{p+q-1} \theta^{p-q+1}] \rightarrow [\lambda^{p+q} \theta^{p-q}])} \quad (64)$$

In other words, $E_1^{p,q} = H(Q_{L+})|_{\lambda^{p+q} \theta^{p-q}}$. On the second page:

$$E_2^{p,q} = H(Q_{L-}, H(Q_{L+}))|_{\lambda^{p+q} \theta^{p-q}} \quad (65)$$

We will be calculating the differentials d_1 and d_2 :

$$d_1 : E_1^{p,q} \rightarrow E_1^{p+1,q} \quad (66)$$

$$d_2 : E_2^{p,q} \rightarrow E_2^{p+2,q-1} \quad (67)$$

Schematically:

$$d_1 : ([\lambda^m \theta^n] + \dots) \rightarrow ([\lambda^{m+1} \theta^{n+1}] + \dots) \quad (68)$$

$$d_2 : ([\lambda^m \theta^n] + \dots) \rightarrow ([\lambda^{m+1} \theta^{n+3}] + \dots) \quad (69)$$

4.3 Structure of $H(Q_{L+})$

Let us denote:

$${}_d\{\lambda \overset{0}{\cup} \theta\}_c = {}_d\{\lambda \cup \theta\}_c - \frac{1}{4}\omega_{dc}||\lambda \cup \theta|| \quad (70)$$

— this is the “ ω -traceless part”, *i.e.* $\omega^{dc}{}_d\{\lambda \overset{0}{\cup} \theta\}_c = 0$.

The cohomology group $H(Q_{L+})$ is generated by the following elements:

$$W^{\alpha\beta} = {}^\alpha\{\theta \overset{0}{\cap} \lambda\}^\beta \quad (71)$$

$$W_{ab} = {}_a\{\theta \overset{0}{\cup} \lambda\}_b \quad (72)$$

$$W_a^\alpha = \theta^\alpha \cap \lambda \cup \theta_a \quad (73)$$

$$Z_b^\alpha = \theta^\alpha \cap \lambda \cup \theta \cap \lambda \overset{0}{\cup} \theta_b \quad (74)$$

$$Z^{\alpha\beta} = \theta^\alpha \cap \lambda \cup \theta \cap \theta \cup \lambda \cap \theta^\beta \quad (75)$$

$$Z_{ab} = \theta_a \cup \lambda \cap \theta \cup \theta \cap \lambda \cup \theta_b \quad (76)$$

$$S = ||\theta \cup \lambda \cap \theta \cup \theta \cap \lambda \cup \theta \cap \lambda \overset{0}{\cup} \theta|| \quad (77)$$

In the Γ -matrix notations:

$$W^{\alpha\beta} \mapsto (\theta \Gamma_{\mathbf{a}} \lambda) \quad (78)$$

$$W_{ab} \mapsto (\theta \Gamma_{\mathbf{s}} \lambda) \quad (79)$$

$$3W_a^\alpha \simeq (\theta \cap \{\lambda \overset{0}{\cup} \theta\} + \{\theta \overset{0}{\cap} \lambda\} \cup \theta) \mapsto \widehat{F}\Gamma_m \theta (\lambda \Gamma^m \theta) \quad (80)$$

$$Z_b^\alpha \simeq \mapsto (\lambda \Gamma^m \theta) (\lambda \Gamma^n \theta) \Gamma_{mn} \theta \quad (81)$$

$$Z^{\alpha\beta} \simeq \mapsto (\lambda \Gamma^m \theta) (\lambda \Gamma^n \theta) (\theta \Gamma_{\mathbf{a}mn} \theta) \quad (82)$$

$$Z_{ab} \simeq \mapsto (\lambda \Gamma^m \theta) (\lambda \Gamma^n \theta) (\theta \Gamma_{\mathbf{s}mn} \theta) \quad (83)$$

$$S \mapsto (\lambda \Gamma^k \theta) (\lambda \Gamma^m \theta) (\lambda \Gamma^n \theta) (\theta \Gamma_{kmn} \theta) \quad (84)$$

In Eqs. (78) and (79) the subindices **a** and **s** enumerate the tangent space to $AdS_5 \times S^5$, namely **a** enumerates the tangent space to AdS_5 and **s** the tangent space to S^5 . (Therefore both **a** and **s** run from 1 to 5.)

4.4 Multiplication in $H(Q_{L+})$

Lemma Suppose X_{dc} is an antisymmetric ω -less rank 2 tensor (*i.e.* $X_{dc} = -X_{cd}$ and $\omega^{dc}X_{dc} = 0$) and Ψ_b is a rank 1 tensor. Then:

$$\Psi_b X_{dc} = -\frac{1}{5}\Psi \cap X_{\bullet b} \omega_{dc} - \frac{4}{5}\Psi \cap X_{\bullet [d} \omega_{c]b} + P_{b[dc]} \quad (85)$$

where

$$\omega^{bd}P_{b[dc]} = \omega^{bd}P_{c[bd]} = 0 \quad (86)$$

Taking this and similar linear-algebraic identities into account, we derive the following multiplication table in $H(Q_{L+})$:

$$W_a^\alpha W_{bc} = -\frac{1}{5}Z_a^\alpha \omega_{bc} - \frac{4}{5}Z_{[b}^\alpha \omega_{c]a} \quad (87)$$

$$W_a^\alpha W_b^\beta = \frac{1}{4}\omega_{ab}Z^{\alpha\beta} + \frac{1}{4}\omega^{\alpha\beta}Z_{ab} \quad (88)$$

$$W_a^\alpha Z_b^\beta = \frac{1}{16}\omega_{ab}\omega^{\alpha\beta}S \quad (89)$$

The other products are zero.

4.5 The meta-symmetry

The “meta-symmetry” is flipping all Greek letters with the corresponding Latin letters, upper indices with lower indices, and \cup with \cap . It is useful to keep track of whether the expression is meta-odd or meta-even. Notice that Q_{L+} is meta-even, and Q_{L-} is meta-odd.

5 The (2,0) part

We will investigate the cohomology of Q_L in the ghost numbers $(p, 0)$ using the spectral sequence of $Q_{L+} + Q_{L-}$. We will find that the cohomology is trivial for all p .

5.1 Cohomology of Q_{L+}

Let us start by looking at the cohomology of Q_{L+} . There are classes of the types:

$$\lambda^0 \theta^0 : (\omega^{\bullet\bullet} \otimes \omega_{\bullet\bullet})^{\otimes(n+1)} \quad (90)$$

$$\lambda \theta : \bullet \{ \theta \overset{0}{\cap} \lambda \} \bullet \otimes (\omega^{\bullet\bullet})^{\otimes n} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} \pm (\text{meta-flip}) \quad (91)$$

(two classes of the type $\lambda \theta$, one meta-odd and one meta-even)

$$\lambda^2 \theta^4 : \bullet \theta \cup \lambda \cap \theta \cup \theta \cap \lambda \cup \theta \bullet \otimes (\omega_{\bullet\bullet})^{\otimes n} \otimes (\omega^{\bullet\bullet})^{\otimes(n+1)} \pm (\text{meta-flip}) \quad (92)$$

(two classes of the type $\lambda^2 \theta^4$, one meta-odd and one meta-even)

$$\lambda^3 \theta^5 : || \theta \cap \lambda \cup \theta \cap \theta \cup \lambda \cap \theta \cup \{ \lambda \cap \theta \} || (\omega^{\bullet\bullet})^{\otimes(n+1)} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} \quad (93)$$

Notice that $\lambda^2 \theta^3$ is missing because of the wrong quantum numbers.

5.2 Higher differentials

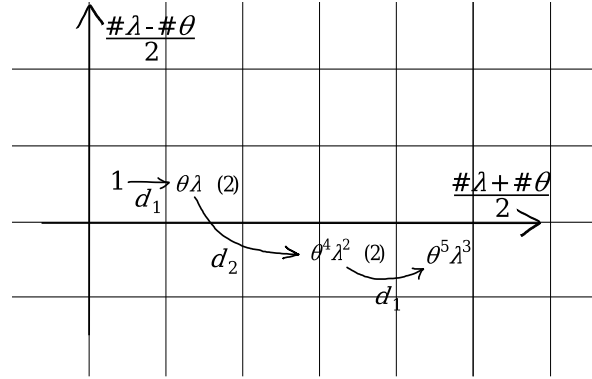


Figure 1: Higher differentials on pages E_1 and E_2

5.2.1 The meta-odd class of the type $\lambda \theta$ is cancelled by d_1 of $\lambda^0 \theta^0$

This follows immediately from (90) and (59).

5.2.2 Two classes of the type $\lambda^2\theta^4$: Φ_0^+ and Φ_0^-

There are two cohomology classes⁵ of Q_{L+} . Let us consider *e.g.* $n = 1$:

$$\begin{aligned} (\Phi_0^\pm)^{\alpha\beta\gamma\delta}_{abcd} &= \frac{1}{4} \theta_{[a} \cup \lambda \cap \theta \cup \theta \cap \lambda \cup \theta_{b]} \omega^{\alpha\beta} \omega^{\gamma\delta} \omega_{cd} \pm \\ &\pm \frac{1}{4} \theta^{[\alpha} \cap \lambda \cup \theta \cap \theta \cup \lambda \cup \theta^{\beta]} \omega_{ab} \omega^{\gamma\delta} \omega_{cd} \end{aligned} \quad (94)$$

In the notations of Section 4.2, this means that $E_1^{3,-1}$ is two-dimensional, generated by Φ_0^\pm . Notice that in Q_{L+} cohomology Φ_0^+ is equivalent to:

$$(\Phi_0)^{\alpha\beta\gamma\delta}_{abcd} = (\theta \cap \lambda \cup \theta)_{[a}^{[\alpha} (\theta \cap \lambda \cup \theta)_{b]}^{\beta]} \omega^{\gamma\delta} \omega_{cd} \quad (95)$$

It is easy to check that:

$$Q_{L+}\Phi_0^+ = Q_{L+}\Phi_0^- = 0 \quad (96)$$

Next we have to calculate the action of Q_{L-} . It will turn out that Φ_0^- is not annihilated by d_1 , and Φ_0^+ is in the image of d_2 . Therefore there is no cohomology in the ghost number $(2, 0)$.

5.2.3 The class of the type $\lambda^3\theta^5$ is cancelled by d_1 of Φ_0^-

We will see that:

- Φ_0^+ is annihilated by d_1
- But Φ_0^- is not:

$$d_1\Phi_0^- = [\lambda^3\theta^5] \quad (97)$$

Therefore $d_1\Phi_0^-$ cancels $||\theta \cap \lambda \cup \theta \cap \theta \cup \lambda \cap \theta \cup \{\lambda \cap \theta\}|| \omega \cdots \omega \in E_1^{4,-1}$

Let us start with Φ_0^+ . Consider the case $n = 1$, the general case is completely analogous. We observe that $Q_{L-}\Phi_0^+$ is not literally zero:

$$\epsilon Q_{L-}\Phi_0^+ = (\theta \cap \lambda \cup \theta) \otimes (\theta \cap \lambda \cup \theta) \otimes ([\theta \cap \epsilon \lambda] \otimes \omega_{\bullet\bullet} + \omega^{\bullet\bullet} \otimes [\epsilon \lambda \cup \theta]) \quad (98)$$

Up to Q_{L+} -exact terms this can be rewritten in the following way:

$$\begin{aligned} (Q_{L-}\Phi_0^+)^{\alpha\beta\gamma\delta}_{abcd} &= \theta^\alpha \cap \lambda \cup \theta_a \otimes \theta_b \cup \lambda \cap \theta^\beta (-\{\theta \cap \lambda\}^{\gamma\delta} \omega_{cd} + \omega^{\gamma\delta} \{\theta \cup \lambda\}_{cd}) + \\ &+ Q_{L+}(\dots) \end{aligned} \quad (99)$$

⁵Notice that $\lambda^2\theta^3$ does not intertwine properly

This is Q_{L+} -equivalent to:

$$\frac{1}{4}\theta^{[\alpha}\cap\lambda\cup\theta\cap\theta\cup\lambda\cap\theta^{\beta]}\omega_{ab}(\{\theta\cap\lambda\}^{\gamma\delta}\omega_{cd}-\omega^{\gamma\delta}\{\theta\cup\lambda\}_{cd})+ \quad (100)$$

$$+\frac{1}{4}\theta_{[a}\cup\lambda\cap\theta\cup\theta\cap\lambda\cup\theta_{b]}\omega^{\alpha\beta}(\{\theta\cap\lambda\}^{\gamma\delta}\omega_{cd}-\omega^{\gamma\delta}\{\theta\cup\lambda\}_{cd}) \quad (101)$$

This is Q_{L+} -equivalent to:

$$\begin{aligned} & \frac{1}{4}\theta^{[\alpha}\cap\lambda\cup\theta\cap\theta\cup\lambda\cap\theta^{\beta]}\omega_{ab}\{\theta\cap\lambda\}^{\gamma\delta}\omega_{cd}- \\ & -\frac{1}{4}\theta_{[a}\cup\lambda\cap\theta\cup\theta\cap\lambda\cup\theta_{b]}\omega^{\alpha\beta}\omega^{\gamma\delta}\{\theta\cup\lambda\}_{cd} \end{aligned} \quad (102)$$

Now let us fuse $\theta\lambda\theta\theta\lambda\theta$ with $\{\theta\lambda\}$. We get:

$$\begin{aligned} & \frac{1}{4}\theta^{[\alpha}\cap\lambda\cup\theta\cap\theta\cup\lambda\cap\theta^{\beta]}\omega_{ab}\{\theta\cap\lambda\}^{\gamma\delta}\omega_{cd}- \\ & -\frac{1}{4}\theta_{[a}\cup\lambda\cap\theta\cup\theta\cap\lambda\cup\theta_{b]}\omega^{\alpha\beta}\omega^{\gamma\delta}\{\theta\cup\lambda\}_{cd}= \\ & =\frac{1}{16}||\theta\cap\lambda\cup\theta\cap\theta\cup\lambda\cap\theta\cup\{\lambda\cap\theta\}||\times \\ & \times (\omega^{[\delta][\alpha}\omega^{\beta]}\omega^{\gamma]}\omega_{ab}\omega_{cd}-\omega^{\alpha\beta}\omega^{\gamma\delta}\omega_{[d][a}\omega_{b]}\omega_{c]})+Q_{L+}(\dots) \end{aligned} \quad (103)$$

One can see that this vanishes after the symmetrization of (ac) and (bd) .

For Φ_0^- the relative sign of the two $\omega\omega\omega\omega$ terms is opposite compared to (103), and therefore $d_1\Phi_0^- \neq 0$.

5.2.4 Φ_0^+ is cancelled by d_2 of the meta-even class of the type $\lambda\theta$

We have so far proven that Φ_0^+ is a nontrivial element of $E_2^{3,-1}$. But we will now show that Φ_0^+ is in the image of d_2 :

$$d_2 : E_2^{1,0} \rightarrow E_2^{3,-1} \quad (104)$$

and therefore Φ_0^+ does not survive on the next page $E_3^{\bullet\bullet}$.

Consider for example the case $n = 1$. The $E_2^{1,0}$ is generated by $A_{abcd}^{\alpha\beta\gamma\delta}$:

$$A = \bullet\{\theta \overset{0}{\cap} \lambda\}\bullet (\omega^{\bullet\bullet})^{\otimes n} (\omega_{\bullet\bullet})^{\otimes(n+1)} \quad (105)$$

We get:

$$(Q_L A) = (n+1) \bullet\{\theta \overset{0}{\cap} \lambda\}\bullet \bullet\{\theta \overset{0}{\cup} \lambda\}\bullet (\omega^{\bullet\bullet} \omega_{\bullet\bullet})^{\otimes n} \quad (106)$$

This is equal to $Q_{L+}C$ where

$$C = \left(\theta \begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \{ \theta \cap \lambda \} \cup \theta \begin{bmatrix} \bullet \\ \bullet \end{bmatrix} + \theta \begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \theta \begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \cap \{ \lambda \cup \theta \} \begin{bmatrix} \bullet \\ \bullet \end{bmatrix} + \right. \\ \left. + \frac{1}{4} \omega^{\bullet\bullet} \theta \begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \cup \theta \cap \{ \lambda \cup \theta \} \begin{bmatrix} \bullet \\ \bullet \end{bmatrix} + \frac{1}{4} \omega_{\bullet\bullet} \begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \{ \theta \cap \lambda \} \cup \theta \cap \theta \begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \right) (\omega^{\bullet\bullet} \omega_{\bullet\bullet})^{\otimes n} \quad (107)$$

(Notice that $\theta_{[a} \cup \{ \theta \cap \lambda \} \cup \theta_{b]}$ is zero.) Let us now calculate $Q_{L-}C$, modulo $\text{Im}(Q_{L+})$. It is useful to represent C in the gamma-matrix notation:

$$C_{as} = (\theta \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma_m \theta) (\theta \Gamma^m \lambda) \otimes \omega \cdots \omega \quad (108)$$

where \mathbf{a} enumerates the tangent space to AdS_5 and \mathbf{s} the tangent space to S^5 . (Therefore both \mathbf{a} and \mathbf{s} run from 1 to 5.) We have to act on this by Q_{L-} and discard the terms which are Q_{L+} -exact. The Q_{L+} cohomology $\theta^4 \lambda^2$ has the following quantum numbers under $so(5) \oplus so(5)$: either vector under the first $so(5)$ and the scalar under the second $so(5)$ or vice versa. We can simply throw away everything else. In particular, when acting on C with Q_{L-} we can throw away the terms arising from the action by the $\theta \lambda \theta \frac{\partial}{\partial \theta}$ in Q_{L-} — see (60), because they do not change the $so(5) \oplus so(5)$ quantum numbers. The other terms are contractions with $\theta \cap \lambda$ and $\lambda \cup \theta$. We split:

$$\theta \cap \lambda = -\frac{1}{4} ||\theta \cap \lambda|| \omega^{\bullet\bullet} + \theta \overset{0}{\cap} \lambda \quad (109)$$

and similarly $\lambda \cup \theta$. The contraction with the first term $-\frac{1}{4} ||\theta \cap \lambda|| \omega^{\bullet\bullet}$ does not change the $so(5) \oplus so(5)$ quantum numbers, and therefore can be discarded. What remains is the rotation with the ω -less part $\theta \overset{0}{\cap} \lambda$, which is $(\theta \Gamma_{\mathbf{a}} \lambda)$ in the Γ -matrix notations, and the $\theta \overset{0}{\cup} \lambda$, which is $(\theta \Gamma_{\mathbf{s}} \lambda)$. Let us look at this rotation from the following point of view. Consider S^5 embedded into \mathbf{R}^6 , and AdS_5 into \mathbf{R}^{2+4} . Therefore ω represents a unit vector in \mathbf{R}^6 , and a unit vector in \mathbf{R}^{2+4} . What kind of an object is C of (107)? It is a product a traceless symmetric tensor $(\mathbf{R}^{2+4})_{\text{Symm},0}^{\otimes(n+1)}$ and a traceless symmetric tensor $(\mathbf{R}^6)_{\text{Symm},0}^{\otimes(n+1)}$. Let us concentrate on the S^5 part of the rotation, *i.e.* on the lower (latin) indices of C . We have:

$$C = V \otimes \omega^{\otimes n} - \text{traces} \quad (110)$$

where V is the S^5 part of $(\theta \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma_m \theta) (\theta \Gamma^m \lambda)$. The vector $Y = (\theta \Gamma_{\mathbf{s}} \lambda)$ is tangent to the S^5 at the point ω . We need to rotate C by the infinitesimal

rotation corresponding to the bivector $Y \wedge \omega$. The rotated tensor $(Y \wedge \omega).C$ is, again, a symmetric and traceless tensor of $so(6)$. It can be represented as a sum of a term proportional to $\omega^{\otimes(n+1)}$, and other terms which are orthogonal to $\omega^{\otimes(n+1)}$. We observe that the term proportional to $\omega^{\otimes(n+1)}$ in $(Y \wedge \omega).C$ equals to $-(Y, V) \omega^{\otimes(n+1)}$. Therefore:

$$(Y, V) \neq 0 \text{ implies } (Y \wedge \omega).C \neq 0 \quad (111)$$

The scalar product (Y, V) where $Y = (\theta \Gamma_s \lambda)$ and $V = (\theta \Gamma_a \Gamma_s \Gamma_m \theta)(\theta \Gamma^m \lambda)$ equals⁶:

$$(Y, V) = (\theta \Gamma^s \lambda)(\theta \Gamma_a \Gamma_s \Gamma_m \theta)(\theta \Gamma^m \lambda) \quad (112)$$

This is a nontrivial class of the Q_{L+} cohomology, Notice that the index **a** remained from the AdS part of C , so this is $(Q_{L-}C)_a$. The $(Q_{L-}C)_s$ is given by a similar expression. The expression (112) is Q_{L+} -equivalent to:

$$\theta^{[\alpha} \cap \lambda \cup \theta \cap \theta \cup \lambda \cap \theta^{\beta]} \quad (113)$$

This means that $d_2 A$ is in fact identified with the Φ_0 , as given by Eq. (94):

$$\frac{1}{4} \theta_{[\bullet} \cup \lambda \cap \theta \cup \theta \cap \lambda \cup \theta_{\bullet]} (\omega^{\bullet\bullet})^{\otimes(n+1)} (\omega_{\bullet\bullet})^{\otimes n} + \quad (114)$$

$$+ \frac{1}{4} \theta^{[\bullet} \cap \lambda \cup \theta \cap \theta \cup \lambda \cap \theta^{\bullet]} (\omega^{\bullet\bullet})^{\otimes n} (\omega_{\bullet\bullet})^{\otimes(n+1)} \quad (115)$$

(The relative sign of the two terms is such that this is meta-even, since (107) is meta-odd.)

5.2.5 Conclusion: the cohomology of Q_L in expressions which do not contain μ is trivial

Indeed, we have seen that all the cohomology classes of Q_{L+} cancel when we correct $Q_{L+} \rightarrow Q_L = Q_{L+} + Q_{L-}$. This shows that the (2,0) part of the vertex can always be gauged away. (And the same is true about the (0,2) part.)

⁶When the index **s** is contracted we assume the summation over the *five* indices enumerating the tangent space of S^5

6 The (1,1) part

6.1 Notations and summary

In this section we will show that $Q = Q_L + Q_R$ has nontrivial cohomology in the ghost number (1,1).

We will introduce the right pure spinor which will be denoted μ . As in Section 4.1.1 we will write:

$$\mu = \begin{pmatrix} 0 & \mu_+ \\ \mu_- & 0 \end{pmatrix} \quad (116)$$

but now there is a minus sign in the expression relating μ_- to μ_+ , compared to Eq. (47):

$$(\mu_-)_\alpha^a = -i\omega^{aa'}(\mu_+)_{a'}^{\alpha'}\omega_{\alpha'\alpha} \quad (117)$$

This is because $\mu \in \mathfrak{g}_1$ while $\lambda \in \mathfrak{g}_3$. This leads to the overall minus sign in the action of Q_{R-} , compared to Eq. (59):

$$\epsilon Q_{R-}\Phi = -\theta \cap \epsilon \mu \cup \Phi - \Phi \cap \epsilon \mu \cup \theta + \left((\theta \cap \epsilon \mu \cup \theta) \frac{\partial}{\partial \theta} \right) \Phi \quad (118)$$

6.2 The cohomology of Q_{L+} on expressions linear in λ and μ

The cohomology of Q_{L+} in the sector of functions linear in both λ and μ is generated by the following expressions:

6.2.1 Two-trace combination

This is the part with the maximal number of ω 's:

$$\overset{\text{tr}}{\Psi}(\theta) = ||\mu \cap \theta \cup \lambda \cap \theta|| (\omega_{\bullet\bullet} \otimes \omega^{\bullet\bullet})^{\otimes(n+1)} \quad (119)$$

This is meta-odd. In the Gamma-matrix notations this corresponds to $(\mu \Gamma_m \theta)(\lambda \Gamma^m \theta)$.

6.2.2 One-trace combinations

There are two of them:

$$\overset{\text{dn}}{\Psi}(\theta) = \left(\mu_{[\bullet} \cup \theta \cap \lambda \cup \theta_{\bullet]} - \frac{1}{4} \omega_{\bullet\bullet} \|\mu \cup \theta \cap \lambda \cup \theta\| \right) (\omega_{\bullet\bullet})^{\otimes n} (\omega^{\bullet\bullet})^{\otimes (n+1)} \quad (120)$$

$$\overset{\text{up}}{\Psi}(\theta) = \left(\mu^{[\bullet} \cap \theta \cup \lambda \cap \theta^{\bullet]} - \frac{1}{4} \omega^{\bullet\bullet} \|\mu \cap \theta \cup \lambda \cap \theta\| \right) (\omega_{\bullet\bullet})^{\otimes (n+1)} (\omega^{\bullet\bullet})^{\otimes n} \quad (121)$$

They are interchanged by the meta-symmetry:

$$\text{meta}(\overset{\text{dn}}{\Psi}) = \overset{\text{up}}{\Psi} \quad , \quad \text{meta}(\overset{\text{up}}{\Psi}) = \overset{\text{dn}}{\Psi} \quad (122)$$

In the Gamma-matrix notations they correspond to $(\mu \hat{F} \Gamma_s \Gamma_m \theta)(\theta \Gamma^m \lambda)$ and $(\mu \hat{F} \Gamma_a \Gamma_m \theta)(\theta \Gamma^m \lambda)$.

6.2.3 Double-traceless combination

The traceless combination is:

$$\begin{aligned} \overset{0}{\Psi}(\theta) = & \left(\mu_{[\bullet}^{[\bullet} \theta^{\bullet]} \cap \lambda \cup \theta_{\bullet]} \right. \\ & + \frac{1}{4} \mu_{[\bullet} \cup \theta \cap \lambda \cup \theta_{\bullet]} \omega^{\bullet\bullet} - \frac{1}{4} \mu^{[\bullet} \cap \theta \cup \lambda \cap \theta^{\bullet]} \omega_{\bullet\bullet} + \\ & \left. + \frac{1}{16} \omega_{\bullet\bullet} \omega^{\bullet\bullet} \|\mu \cap \theta \cup \lambda \cap \theta\| \right) \otimes (\omega^{\bullet\bullet} \otimes \omega_{\bullet\bullet})^{\otimes n} \end{aligned} \quad (123)$$

This is meta-odd. In the Gamma-matrix notations this is $(\mu \Gamma_a \Gamma_s \Gamma_m \theta)(\theta \Gamma^m \lambda)$.

What happens when we act by Q_{R+} It turns out that both $Q_{R+} \overset{\text{up}}{\Psi}$ and $Q_{R+} \overset{\text{dn}}{\Psi}$ are nontrivial in the Q_{L+} -cohomology, while $Q_{R+} \overset{\text{tr}}{\Psi} = 0$ and $Q_{R+} \overset{0}{\Psi}$ is trivial in the Q_{L+} -cohomology.

What happens when we act by Q_{L-} There is only one obstacle at $[\mu \lambda^2 \theta^4]$ for constructing the full Q_L -closed expression. Therefore there is a linear combination of $\overset{0}{\Psi}$ and $\overset{\text{tr}}{\Psi}$ that can be extended to the full Q_L -closed expression.

6.3 Q_{L-} on $\mu \otimes \theta \lambda \theta \otimes \omega$'s

Let us consider the following expression:

$$\Psi(\theta) = \mu \begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \theta \begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \cap \lambda \cup \theta \begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \otimes (\omega^{\bullet\bullet})^{\otimes n} \otimes (\omega_{\bullet\bullet})^{\otimes n} \quad (124)$$

Notice that this is meta-odd. Therefore $Q_{L-}\Psi$ is meta-even. This, in particular, implies, that when we calculate the action of Q_{L-} on Ψ , the sum of terms arising from the action of Q_{L-} on ω are all Q_{L+} -exact. Indeed, these terms would have the form $\mu \otimes (\theta \lambda \theta) \otimes Q_{L-}(\cup \otimes \cap)$. The fusion of $\theta \lambda \theta$ and $Q_{L-}(\cup \otimes \cap)$ produces a meta-even expression. But the cohomology obstacle, which is the cohomology class of Q_{L+} , is of the form $\theta \cap \lambda \cup \theta \cap \lambda \cup \theta$, and is meta-odd. Therefore, we can neglect the terms which appear when Q_{L-} acts on the ω s.

Also observe that $Q_{L-}\theta^{\bullet} \cap \lambda \cup \theta_{\bullet} = 0$. Therefore the only nontrivial contribution arises when Q_{L-} acts on μ :

$$(\mu^{[\alpha} \cap \lambda \cup \theta_{[a} - \theta^{[\alpha} \cap \lambda \cup \mu_{[a}] \theta^{\beta]} \cap \lambda \cup \theta_{b]}) \quad (125)$$

This is meta-even, and can be rewritten modulo Q_{L+} -exact terms as follows:

$$\frac{1}{2} \theta^{[\alpha} \cap \lambda \cup \theta_{[a} \{ \theta \cup \lambda \} \cap \mu^{\beta]} + \frac{1}{2} \theta_{[a} \cup \lambda \cap \theta^{\alpha} \{ \theta \cap \lambda \} \cup \mu_{b]} \quad (126)$$

Given the multiplication rule (87), this is equivalent in the Q_{L+} -cohomology to the following expression:

$$\begin{aligned} & -\frac{1}{10} \theta^{[\alpha} \cap \lambda \cup \theta \cap \{ \theta \overset{0}{\cup} \lambda \} \cap \mu^{\beta]} \omega_{ab} - \frac{2}{5} \theta^{[\alpha} \cap \lambda \cup \theta \cap \{ \theta \overset{0}{\cup} \lambda \}_{[a} \mu_{b]}^{\beta]} - \\ & -\frac{1}{10} \theta_{[a} \cup \lambda \cap \theta \cup \{ \theta \overset{0}{\cap} \lambda \} \cup \mu_{b]} \omega^{\alpha\beta} - \frac{2}{5} \theta_{[a} \cup \lambda \cap \theta \cup \{ \theta \overset{0}{\cap} \lambda \}^{[\alpha} \mu_{b]}^{\beta]} \quad (127) \end{aligned}$$

Notice that the two terms which have both indices of μ uncontracted cancel each other modulo Q_{L+} -exact terms. Therefore:

$$\begin{aligned} & Q_{L-} \left(\mu \begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \theta \begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \cap \lambda \cup \theta \begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \otimes (\omega^{\bullet\bullet})^{\otimes n} \otimes (\omega_{\bullet\bullet})^{\otimes n} \right) = \\ & = -\frac{1}{10} \theta \begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \cap \lambda \cup \theta \cap \{ \theta \overset{0}{\cup} \lambda \} \cap \mu \begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \otimes (\omega^{\bullet\bullet})^{\otimes n} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} - \\ & \quad - \frac{1}{10} \theta \begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \cup \lambda \cap \theta \cup \{ \theta \overset{0}{\cap} \lambda \} \cup \mu \begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \otimes (\omega^{\bullet\bullet})^{\otimes(n+1)} \otimes (\omega_{\bullet\bullet})^{\otimes n} + \\ & \quad + Q_{L+}(\text{smth}) \quad (128) \end{aligned}$$

Comment: relating $Q_{L-}(\mu \otimes \theta\lambda\theta)$ to $Q_{R+}(\theta\lambda\theta \otimes \theta\lambda\theta)$ Since we have shown that we can neglect the terms where Q_{L-} is hitting ω 's, we get:

$$\begin{aligned} & Q_{R+} \left(\theta^{[\alpha} \cap \lambda \cup \theta_{[a} \theta_{b]} \cup \lambda \cap \theta^{\beta]} \omega^{\gamma\delta} \omega_{cd} \right) = \\ & = Q_{L-} \left(\mu_{[a}^{[\alpha} \theta_{b]} \cup \lambda \cap \theta^{\beta]} \omega^{\gamma\delta} \omega_{cd} \right) \end{aligned} \quad (129)$$

6.4 Q_{L-} on $\theta\lambda\theta\mu \otimes \omega$'s

Let us now calculate this:

$$Q_{L-} \left(\theta_{[a} \cup \lambda \cap \theta \cup \mu_{b]} \omega^{\alpha\beta} \omega^{\gamma_1\delta_1} \dots \omega^{\gamma_n\delta_n} \omega_{c_1d_1} \dots \omega_{c_nd_n} \right) \quad (130)$$

Comment on notations: It turns out that the structure of the formulas has some regular dependence on n . It is enough to consider the case $n = 1$, and then put the coefficient n where necessary. To save space we will replace:

$$\omega^{\gamma_1\delta_1} \dots \omega^{\gamma_n\delta_n} \omega_{c_1d_1} \dots \omega_{c_nd_n} \longrightarrow \omega^{\gamma\delta} \omega_{cd} \quad (131)$$

but keep track of the coefficient n . We have:

$$Q_{L-} \left(\theta_{[a} \cup \lambda \cap \theta \cup \mu_{b]} \omega_{cd} \omega^{\alpha\beta} \omega^{\gamma\delta} \right) = \quad (132)$$

$$= \theta_{[a} \cup \lambda \cap \theta \cup \lambda \cap \theta \cup \mu_{b]} \omega_{cd} \omega^{\alpha\beta} \omega^{\gamma\delta} + \quad (133)$$

$$+ \theta_{[a} \cup \lambda \cap \theta \cup \mu \cap \lambda \cup \theta_{b]} \omega_{cd} \omega^{\alpha\beta} \omega^{\gamma\delta} + \quad (134)$$

$$+ n \theta_{[a} \cup \lambda \cap \theta \cup \mu_{b]} \omega^{\alpha\beta} \omega^{\gamma\delta} \{ \lambda_c \cup \theta_d \} - \quad (135)$$

$$- (n+1) \theta_{[a} \cup \lambda \cap \theta \cup \mu_{b]} \omega_{cd} \{ \lambda^\alpha \cap \theta^\beta \} \omega^{\gamma\delta} \quad (136)$$

We use the multiplication rule (87) to transform this expression. After a calculation, we get up to Q_{L+} -exact terms:

$$Q_{L-} \left(\theta_{[a} \cup \lambda \cap \theta \cup \mu_{b]} \omega_{cd} \omega^{\alpha\beta} \omega^{\gamma\delta} - \quad (137)$$

$$- \theta^{[\alpha} \cap \lambda \cup \theta \cap \mu^{\beta]} \omega^{\gamma\delta} \omega_{ab} \omega_{cd} \right) = \quad (138)$$

$$= -\frac{1}{5} (4n+3) \left(\mu_{[a} \cup \theta \cap \lambda \cup \theta \cap \lambda \overset{0}{\cap} \theta_{b]} \omega_{cd} \omega^{\alpha\beta} \omega^{\gamma\delta} + \quad (139)$$

$$+ \mu^{[\alpha} \cap \theta \cup \lambda \cap \theta \cup \lambda \overset{0}{\cap} \theta^{\beta]} \omega^{\gamma\delta} \omega_{ab} \omega_{cd} \right) \quad (140)$$

Notice that the meta-odd pieces cancelled, and we are left on the RHS with the meta-even expression; this is because the LHS we have Q_{L-} of the meta-odd expression.

6.4.1 Acting on the double trace

Again, the dependence on n is regular. We consider first the case $n = 0$:

$$\begin{aligned}
Q_{L-} (||\mu \cap \theta \cup \lambda \cap \theta|| \omega^{\alpha\beta} \omega_{ab}) &= \\
&= ||\mu \cap \theta \cup \lambda \cap \theta|| ({}_a\{\lambda \cup \theta\}_b \omega^{\alpha\beta} - \omega_{ab} {}^\alpha\{\lambda \cap \theta\}^\beta) = \\
&= - ||\mu \cup \theta \cap \lambda \cup \theta|| {}_a\{\lambda \cup \theta\}_b \omega^{\alpha\beta} - \\
&\quad - ||\mu \cap \theta \cup \lambda \cap \theta|| \omega_{ab} {}^\alpha\{\lambda \cap \theta\}^\beta = \\
&= \frac{4}{5} \mu_{[b} \cup \theta \cap \lambda \cup \theta \cap \{\lambda \overset{0}{\cup} \theta\}_{a]} \omega^{\alpha\beta} + \\
&\quad + \frac{4}{5} \mu^{[\beta} \cap \theta \cup \lambda \cap \theta \cup \{\lambda \overset{0}{\cap} \theta\}^{\alpha]} \omega_{ab}
\end{aligned} \tag{141}$$

For general n :

$$\begin{aligned}
Q_{L-} (||\mu \cap \theta \cup \lambda \cap \theta|| (\omega^{\bullet\bullet})^{\otimes(n+1)} (\omega_{\bullet\bullet})^{\otimes(n+1)}) &= \\
&= - \frac{4}{5} (n+1) \mu_{[\bullet} \cup \theta \cap \lambda \cup \theta \cap \{\lambda \overset{0}{\cup} \theta\}_{\bullet]} \otimes (\omega_{\bullet\bullet})^{\otimes n} \otimes (\omega^{\bullet\bullet})^{\otimes(n+1)} - \\
&\quad - \frac{4}{5} (n+1) \mu^{[\bullet} \cap \theta \cup \lambda \cap \theta \cup \{\lambda \overset{0}{\cap} \theta\}^{\bullet]} \otimes (\omega^{\bullet\bullet})^{\otimes n} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)}
\end{aligned} \tag{142}$$

Let us denote:

$${}^{\text{tr}}\Psi(\theta) = ||\mu \cap \theta \cup \lambda \cap \theta|| (\omega^{\bullet\bullet})^{\otimes(n+1)} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} \tag{143}$$

6.4.2 Acting on the traceless combination with Q_{L-}

The traceless combination is:

$$\overset{0}{\Psi}(\theta)_{ab} = \mu_{[a}^{[\alpha} \theta^{\beta]} \cap \lambda \cup \theta_{b]} \tag{144}$$

$$+ \frac{1}{4} \mu_{[a} \cup \theta \cap \lambda \cup \theta_{b]} \omega^{\alpha\beta} - \frac{1}{4} \mu^{[\alpha} \cap \theta \cup \lambda \cap \theta^{\beta]} \omega_{ab} + \tag{145}$$

$$+ \frac{1}{16} \omega_{ab} \omega^{\alpha\beta} ||\mu \cap \theta \cup \lambda \cap \theta|| \tag{146}$$

Summarizing Eqs. (128), (138) and (142), we get:

$$Q_{L-} \overset{0}{\Psi} = \frac{2n+5}{20} \left(\mu_{[a} \cup \theta \cap \lambda \cup \theta \cap \lambda \overset{0}{\cup} \theta_{b]} \omega^{\alpha\beta} + \right. \tag{147}$$

$$\left. + \mu^{[\alpha} \cap \theta \cup \lambda \cap \theta \cup \lambda \overset{0}{\cap} \theta^{\beta]} \omega_{ab} \right) \tag{148}$$

Expression in the kernel of Q_{L-} Therefore, let us define:

$$\Psi_{\text{complete}} = \frac{20}{2n+5} \overset{0}{\Psi} + \frac{5}{8n+8} \overset{\text{tr}}{\Psi} \quad (149)$$

We get:

$$Q_{L-} \Psi_{\text{complete}} = Q_{L+} [\mu \lambda \theta^4] \quad (150)$$

This means that we can redefine Ψ_{complete} by adding to it the terms of the order θ^4 and higher so that:

$$Q_L \Psi_{\text{complete}} = 0 \quad (151)$$

6.5 Acting with Q_{R+}

6.5.1 Further adjustment of Ψ_{complete}

The so constructed Ψ_{complete} is not annihilated by Q_{R+} . We will now modify Ψ_{complete} by adding to it a Q_L -exact expressions, so that the leading term of the modified Ψ_{complete} is annihilated by Q_{R+} .

We will start with the following modification:

$$\Psi_{\text{complete}} \mapsto \Psi_{\text{complete}} + \frac{20}{2n+5} \times \frac{1}{3} Q_L \left(\mu_{[a}^{[\alpha} \theta^{\beta]} \cap \theta \cup \theta_{b]} + \quad (152)$$

$$+ \frac{1}{4} \mu_{[a} \cup \theta \cap \theta \cup \theta_{b]} \omega^{\alpha\beta} - \frac{1}{4} \mu^{[\alpha} \cap \theta \cup \theta \cap \theta^{\beta]} \omega_{ab} + \quad (153)$$

$$+ \frac{1}{16} \omega_{ab} \omega^{\alpha\beta} ||\mu \cap \theta \cup \theta \cap \theta|| \Big) \quad (154)$$

This only changes in the leading expression of Ψ_{complete} is in the ω -less part $\overset{0}{\Psi}$; the leading ω -less part of the modified Ψ_{complete} is now the following:

$$\begin{aligned} \overset{0}{\Psi} (\theta)_{ab}^{\alpha\beta} &= \frac{20}{2n+5} \times \frac{1}{3} \left(\mu_{[a}^{[\alpha} \theta^{\beta]} \{ \theta \overset{0}{\cap} \lambda \} \cup \theta_{b]} + \mu_{[a}^{[\alpha} \theta^{\beta]} \cap \{ \lambda \overset{0}{\cup} \theta \}_{b]} \right) + \\ &+ (\text{subtraction of } \omega\text{-contractions}) \end{aligned} \quad (155)$$

Observe that in the Γ -matrix notations this modified $\overset{0}{\Psi}$ has the following form:

$$\overset{0}{\Psi}_{\text{as}} \simeq (\mu \Gamma_{\text{a}} \Gamma_{\text{s}} \Gamma_m \theta) (\lambda \Gamma^m \theta) \quad (156)$$

where \simeq means “proportional to”. (The index notations is as explained after Eqs. (78) and (79).) We get:

$$\begin{aligned} Q_{R+} (\mu \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma_m \theta) (\lambda \Gamma^m \theta) &= -2 (\mu \Gamma_{\mathbf{a}} \theta) (\lambda \Gamma_{\mathbf{s}} \mu) + 2 (\mu \Gamma_{\mathbf{s}} \theta) (\lambda \Gamma_{\mathbf{a}} \mu) = \\ &= 2 Q_{L+} (\mu \Gamma_{\mathbf{a}} \theta) (\mu \Gamma_{\mathbf{s}} \theta) = \frac{1}{2} Q_{L+} Q_{R+} (\theta \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma_m \theta) (\mu \Gamma^m \theta) \end{aligned} \quad (157)$$

This implies:

$$Q_{R+} \left((\mu \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma_m \theta) (\lambda \Gamma^m \theta) + \frac{1}{2} Q_{L+} (\theta \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma_m \theta) (\mu \Gamma^m \theta) \right) = 0 \quad (158)$$

The properties of the expression

$$\Psi_{\mathbf{as}}^{0,\text{new}} = (\mu \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma_m \theta) (\lambda \Gamma^m \theta) + \frac{1}{2} Q_{L+} (\theta \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma_m \theta) (\mu \Gamma^m \theta) \quad (159)$$

are studied in Appendix A, where it is shown that this expression is symmetric under the exchange of λ and μ .

Let us modify Ψ_{complete} once more by adding to it the Q_L -exact expression in the following way:

$$\begin{aligned} \Psi_{\text{complete}} &= \Psi_{\text{complete}} + \\ &+ \frac{20}{2n+5} \times \frac{1}{6} Q_L \left(\theta_{[a}^{[\alpha} \theta^{\beta]} \{ \theta \overset{0}{\cap} \mu \} \cup \theta_{b]} + \theta_{[a}^{[\alpha} \theta^{\beta]} \cap \{ \mu \overset{0}{\cup} \theta \}_{b]} - \right. \\ &\quad \left. - \omega\text{-traces} \right) \end{aligned} \quad (160)$$

The leading term of the so modified Ψ_{complete} is in the Γ -matrix notations proportional to (159). Therefore the Q_{R+} on the leading term of the modified Ψ_{complete} is zero.

Now we have:

$$Q_R \Psi_{\text{complete}} = [\mu^2 \lambda \theta^3] \quad (161)$$

$$Q_L Q_R \Psi_{\text{complete}} = 0 \quad (162)$$

Therefore exists $\Phi = [\mu^2 \theta^4] + \dots$:

$$Q_R \Psi_{\text{complete}} + Q_L \Phi = 0 \quad (163)$$

Observe that $Q_L Q_R \Phi = 0$, therefore $Q_R \Phi = 0$, therefore we find a BRST-closed expression:

$$(Q_L + Q_R) (\Psi_{\text{complete}} + \Phi) = 0 \quad (164)$$

Let us denote:

$$v(\lambda, \mu) = \Psi_{\text{complete}} + \Phi \quad (165)$$

Eq. (164) shows that $v(\lambda, \mu)$ is in the kernel of Q .

Comment on the symmetry of Ψ Notice that the leading ω -double-trace part $\overset{\text{tr}}{\Psi}$ is *antisymmetric* under the exchange of $\mu \leftrightarrow \lambda$, while the leading ω -traceless part is symmetric under such an exchange.

Comment on the relation to [7] Notice that $\overset{\text{tr}}{\Psi}$ contains the expression $||\mu \cap \theta \cup \lambda \cap \theta||$, which is Q_L -closed and up to a Q_L -exact expression equal to the dilaton vertex $\text{Str}(\lambda_3 \lambda_1)$ of [7]. It is multiplied by $\omega \cdots \omega$. In terms of the ansatz (3) this corresponds to multiplying the $\text{Str}(\lambda_3 \lambda_1)$ by the (x, θ) -dependent profile wave function of the excitation (while the dilaton of [7] was constant, *i.e.* (x, θ) -independent). However simply taking:

$$\text{Str}(\lambda_3 \lambda_1) \mapsto \text{Str}(\lambda_3 \lambda_1) f(x, \theta) \quad (166)$$

would not be BRST-closed. This is why we needed to add the ω -traceless part.

6.6 Could $v(\lambda, \mu)$ be BRST-exact?

Notice that $v(\lambda, \mu)$ does not have a term quadratic in λ . Therefore, the only way it could be Q -exact is the following:

$$v(\lambda, \mu) = QA(\lambda) + QB(\mu) \quad (167)$$

where $A(\lambda)$ and $B(\mu)$ are linear functions of λ and μ respectively, and moreover $Q_L A(\lambda) = 0$. Then $A(\lambda)$ is Q_L -exact:

$$A(\lambda) = Q_L C \quad (168)$$

because we have proven in Section 5 that the cohomology of Q_L is trivial. Therefore:

$$v(\lambda, \mu) = Q_R Q_L C + QB(\mu) \quad (169)$$

This means that:

$$v(\lambda, \mu)_{1,1} = Q_L(X) \quad (170)$$

where $X = -Q_R C + B$. Notice that $v(\lambda, \mu)_{1,1}$ is of the type $\lambda\mu\theta\theta$ plus terms of the higher order in θ . Since the leading term in $v(\lambda, \mu)_{1,1}$ is nontrivial in Q_{L+} -cohomology, it must be that $X = [\mu\theta] + \dots$ where the coefficient of $[\mu\theta]$ is nonzero; but the expression of the type $[\mu\theta]$ cannot be annihilated by Q_{L+} . This means that the right hand side of (170) cannot be of the type $[\lambda\mu\theta\theta]$, because it would necessarily contain the $[\lambda\mu]$ term (term without θ 's).

Conclusion We therefore conclude that the following expression:

$$v(\lambda, \mu) = \Psi_{\text{complete}} + \Phi \quad (171)$$

is a nontrivial covariant vertex.

7 Gauge choices

7.1 Symmetry properties under the exchange $\lambda \leftrightarrow \mu$

Notice that Q_+ is symmetric with respect to the exchange $\lambda \leftrightarrow \mu$, and Q_- is antisymmetric. Therefore we have the following symmetry of the BRST complex:

$$(Ev)(\lambda, \mu, \theta) = v(\mu, \lambda, i\theta) \quad (172)$$

Our vertex, as we constructed it, is a sum of the $\mu\mu$ part and the $\lambda\mu$ part; the $\lambda\lambda$ part is zero; this is the “asymmetric gauge”.

7.2 Rocket gauge

We can do a gauge transformation removing the $\mu\mu$ part, and get the “rocket gauge” where the vertex is purely $\lambda\mu$. The leading term of such a gauge transformation is:

$$\phi = \bullet\{\mu \overset{0}{\cap} \theta\}^\bullet \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} \otimes (\omega^{\bullet\bullet})^{\otimes n} \quad (173)$$

After this gauge transformation the vertex is:

$$v_{\text{rocket}}|_{\lambda\mu} = \bullet\{\mu \overset{0}{\cap} \lambda\}^\bullet \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} \otimes (\omega^{\bullet\bullet})^{\otimes n} + \dots \quad (174)$$

$$v_{\text{rocket}}|_{\lambda\lambda} = v_{\text{rocket}}|_{\mu\mu} = 0 \quad (175)$$

The leading (*i.e.* θ^0) term of Ψ_{rocket} is symmetric with respect to the exchange $\lambda \leftrightarrow \mu$, the θ^2 term is antisymmetric with respect to $\lambda \leftrightarrow \mu$, then the θ^4 term is again symmetric and so on. The symmetry w.r.to $\lambda \leftrightarrow \mu$ correlates with the power of θ .

7.3 Airplane gauge

Consider:

$$v_{\text{air}} = \frac{1}{2}(v + Ev) \quad (176)$$

where Ev is defined in (172). This is equal to:

$$\begin{aligned} v_{\text{air}} = & \frac{5}{8} \left(\frac{1}{n+1} \Psi^{\text{tr}} + \right. \\ & + \frac{1}{2} \bullet \theta \cap \lambda \cup \theta \cap \theta \cup \lambda \cap \theta^\bullet \otimes (\omega^{\bullet\bullet})^{\otimes n} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} + \\ & + \frac{1}{2} \bullet \theta \cup \lambda \cap \theta \cup \theta \cap \lambda \cup \theta_\bullet \otimes (\omega_{\bullet\bullet})^{\otimes n} \otimes (\omega^{\bullet\bullet})^{\otimes(n+1)} + \\ & + \frac{1}{2} \bullet \theta \cap \lambda \cup \theta \cap \theta \cup \lambda \cap \theta^\bullet \otimes (\omega^{\bullet\bullet})^{\otimes n} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} + \\ & + \frac{1}{2} \bullet \theta \cup \lambda \cap \theta \cup \theta \cap \lambda \cup \theta_\bullet \otimes (\omega_{\bullet\bullet})^{\otimes n} \otimes (\omega^{\bullet\bullet})^{\otimes(n+1)} + \\ & + \frac{1}{2} \bullet \theta \cap \mu \cup \theta \cap \theta \cup \mu \cap \theta^\bullet \otimes (\omega^{\bullet\bullet})^{\otimes n} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} + \\ & + \frac{1}{2} \bullet \theta \cup \mu \cap \theta \cup \theta \cap \mu \cup \theta_\bullet \otimes (\omega_{\bullet\bullet})^{\otimes n} \otimes (\omega^{\bullet\bullet})^{\otimes(n+1)} + \\ & + \frac{1}{2} \bullet \theta \cap \mu \cup \theta \cap \theta \cup \mu \cap \theta^\bullet \otimes (\omega^{\bullet\bullet})^{\otimes n} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} + \\ & + \frac{1}{2} \bullet \theta \cup \mu \cap \theta \cup \theta \cap \mu \cup \theta_\bullet \otimes (\omega_{\bullet\bullet})^{\otimes n} \otimes (\omega^{\bullet\bullet})^{\otimes(n+1)} + \left. \right) \\ & + \dots \end{aligned} \quad (177)$$

where ... mean terms of the higher order in θ , more precisely terms of the form $\mu\lambda\theta^{\geq 4}, \mu\mu\theta^{\geq 6}$ and $\lambda\lambda\theta^{\geq 6}$. Indeed from (142) we get:

$$\begin{aligned} Q_{L-} \left(||\mu \cap \theta \cup \lambda \cap \theta|| (\omega^{\bullet\bullet})^{\otimes(n+1)} (\omega_{\bullet\bullet})^{\otimes(n+1)} \right) = \\ = -\frac{4}{5}(n+1) \mu_{[\bullet} \cup \theta \cap \lambda \cup \theta \cap \{\lambda \overset{0}{\cup} \theta\}_{\bullet]} (\omega^{\bullet\bullet})^{\otimes(n+1)} (\omega_{\bullet\bullet})^{\otimes n} - \\ - \frac{4}{5}(n+1) \mu^{[\bullet} \cap \theta \cup \lambda \cap \theta \cup \{\lambda \overset{0}{\cap} \theta\}^{\bullet]} (\omega^{\bullet\bullet})^{\otimes n} (\omega_{\bullet\bullet})^{\otimes(n+1)} \end{aligned} \quad (178)$$

On the other hand:

$$\begin{aligned} Q_{R+} \left({}^\alpha\theta \cap \lambda \cup \theta \cap \theta \cup \lambda \cap \theta^\beta - \frac{1}{4} \omega^{\alpha\beta} ||\theta \cap \lambda \cup \theta \cap \theta \cup \lambda \cap \theta|| \right) \simeq \\ \simeq \frac{8}{5} \mu^{[\alpha} \cap \{\lambda \overset{0}{\cup} \theta\} \cap \theta \cup \lambda \cap \theta^\beta] - \frac{2}{5} \omega^{\alpha\beta} ||\mu \cap \{\lambda \overset{0}{\cup} \theta\} \cap \theta \cup \lambda \cap \theta|| \end{aligned} \quad (179)$$

$$\begin{aligned} Q_{R+} \left({}_a\theta \cup \lambda \cap \theta \cup \theta \cap \lambda \cup \theta_b - \frac{1}{4} \omega_{ab} ||\theta \cup \lambda \cap \theta \cup \theta \cap \lambda \cup \theta|| \right) \simeq \\ \simeq \frac{8}{5} \mu_{[a} \cup \{\lambda \overset{0}{\cap} \theta\} \cup \theta \cap \lambda \cup \theta_b] - \frac{2}{5} \omega_{ab} ||\mu \cup \{\lambda \overset{0}{\cap} \theta\} \cup \theta \cap \lambda \cup \theta|| \end{aligned} \quad (180)$$

In other words:

$$Q_{R+} (\theta \cap \lambda \cup \theta \cap \theta \cup \lambda \cap \theta)_{\omega-\text{less}}^{\bullet\bullet} = \frac{8}{5} \left(\mu \cap \{\lambda \overset{0}{\cup} \theta\} \cap \theta \cup \lambda \cap \theta \right)_{\omega-\text{less}}^{[\bullet\bullet]} \quad (181)$$

$$Q_{R+} (\theta \cup \lambda \cap \theta \cup \theta \cap \lambda \cup \theta)_{\bullet\bullet}^{\omega-\text{less}} = \frac{8}{5} \left(\mu \cup \{\lambda \overset{0}{\cap} \theta\} \cup \theta \cap \lambda \cup \theta \right)_{[\bullet\bullet]}^{\omega-\text{less}} \quad (182)$$

Therefore v_{air} is BRST-closed; the BRST variation of the “fuselage” $\mu\lambda$ is cancelled by the BRST variations of the “wings” $\mu\mu$ and $\lambda\lambda$.

8 B -field and dilaton for $\Psi = \mathbf{1} \otimes V$

8.1 Simplifications in the special case $\Psi = \mathbf{1} \otimes V$

Using (177) in (3) with $\Psi = \mathbf{1} \otimes V$ and g given by (185) we get, to the lowest order in θ s:

$$V = \langle v(\lambda, \mu) |_{\theta \rightarrow \theta_L + \theta_R}, e^x V \rangle \quad (183)$$

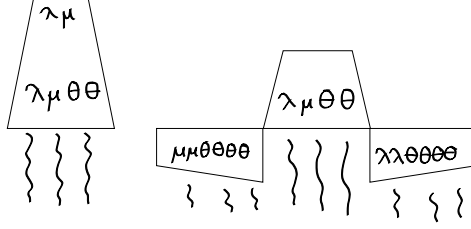


Figure 2: The rocket gauge and the airplane gauge; terms of higher order in θ are shown as wavy lines

The effect of substitution $\theta = \theta_L + \theta_R$ is studied in Appendix A.2, where we show that the leading term of the θ -expansion is proportional to:

$$\langle (\omega^{\bullet\bullet})^{\otimes(n+1)} \otimes (\omega^{\bullet\bullet})^{\otimes(n+1)}, e^x V \rangle (\lambda \Gamma^m \theta_L) (\mu \Gamma_m \theta_R) \quad (184)$$

8.2 Dilaton profile in the flat space limit

8.2.1 Spectral sequence of the flat space expansion

Consider the near flat space expansion with both x and θ scaling like R^{-1} . Let us choose the gauge (cp. (47) and (117)):

$$g = \exp \begin{pmatrix} 0 & \theta_L + \theta_R \\ i(\cap(\theta_L - \theta_R) \cup) & 0 \end{pmatrix} \exp \begin{pmatrix} x_A & 0 \\ 0 & x_S \end{pmatrix} \quad (185)$$

In this gauge the expression for Q up to the order R^{-1} is⁷:

$$Q_{\text{approx}} = \lambda \frac{\partial}{\partial \theta_L} + \mu \frac{\partial}{\partial \theta_R} + ((\theta_L \Gamma^m \lambda) - (\theta_R \Gamma^m \mu)) \frac{\partial}{\partial x^m} \quad (186)$$

The next terms will be of the order R^{-2} , for example $\theta \theta \lambda \partial_\theta$. Observe that with our choice of the gauge (185) there are no terms of the type $x \lambda \frac{\partial}{\partial \theta}$. In other words this approximation of the AdS BRST operator looks in this gauge exactly like the flat space BRST operator. Moreover, let us consider the following splitting of Q :

$$Q = \lambda \frac{\partial}{\partial \theta_L} + \mu \frac{\partial}{\partial \theta_R} + Q_1 \quad (187)$$

⁷Notice that there is a minus sign in front of $(\theta_R \Gamma^m \mu)$. This means that the flat space limit θ_R and λ_R are actually $i\theta_R$ and $i\mu$.

where the first term in the expansion of Q_1 is the $((\theta_L \Gamma^m \lambda) - (\theta_R \Gamma^m \mu)) \frac{\partial}{\partial x^m}$ on the right hand side of (186). Observe that the gauge (185) leads to a natural grading. Namely, let F^p denote the space of functions of $(\lambda, \mu, x, \theta_L, \theta_R)$ having the degree in λ , plus the degree in μ , plus the degree in θ_L , plus the degree in θ_R , greater or equal to $2p$. We observe the following action of operators on grading:

$$\lambda \frac{\partial}{\partial \theta_L} + \mu \frac{\partial}{\partial \theta_R} \quad : \quad F^p \rightarrow F^p \quad (188)$$

$$Q_1 \quad : \quad F^p \rightarrow F^{p+1} \quad (189)$$

This is special to the gauge (185); in this sense this a good gauge choice for the near flat space expansion. Let us calculate the cohomology of Q using the spectral sequence of this filtration. The first page $E_1^{p,q}$ is:

$$E_1^{p,q} = H \left(\lambda \frac{\partial}{\partial \theta_L} + \mu \frac{\partial}{\partial \theta_R}, F^p \right) \Big|_{\text{degree in } \lambda + \text{degree in } \mu = p+q} \quad (190)$$

and the first differential $d_1 : E_1^{p,q} \rightarrow E_1^{p+1,q}$ is induced by Q_1 .

We see from (186) that d_1 acts like in flat space. In particular, this shows that the vertex which we constructed is not Q -exact. Indeed, observe that $E_1^{p,q} = 0$ when $p+q = 1$ and $p < 1$ (since $q = \frac{1}{2}(\text{gh}\# - \#\theta)$). This implies that $E_1^{2,0}$ can only be cancelled by $d_1(E_1^{1,0})$ and not by any higher differential $d_{>1}$. But d_1 acts as in flat space. We will see now that for $n \geq 1$ the flat space limit of our vertex corresponds to a nontrivial dilaton profile. This means that it cannot be gauged away in flat space and therefore neither in AdS.

8.2.2 Polynomial SUGRA solutions in flat space

To get the flat space limit we expand (184) in powers of x and keep the lowest order terms in x . This results in expressions of the type:

$$P(x_A, x_S) (\lambda \Gamma^m \theta_L) (\mu \Gamma_m \theta_R) \quad (191)$$

where $P(x_A, x_S)$ are harmonic homogeneous polynomials of x .

Therefore in the flat space limit our states become polynomial in the coordinates. Notice that the linearized solutions most commonly studied in string theory are exponential, of the form e^{ikx} . In our opinion, the polynomial

solutions deserve further investigation. The exponential solutions factorize into the left and right moving parts, $e^{ikx_L}e^{ikx_R}$. The polynomial solutions do not factorize.

8.3 Dilaton profile in $AdS_5 \times S^5$

We obtain $B_{\mu\nu} = 0$ and $G_{\mu\nu} = \phi(x)\delta_{\mu\nu}$ where:

$$\phi(x) = \omega^{\alpha_1\beta_1} \dots \omega^{\alpha_{n+1}\beta_{n+1}} \omega_{a_1b_1} \dots \omega_{a_{n+1}b_{n+1}} (gV)_{\alpha_1\beta_1 \dots \alpha_{n+1}\beta_{n+1}}^{a_1b_1 \dots a_{n+1}b_{n+1}} \quad (192)$$

In this formula $g = e^x$ parametrizes the bosonic space $AdS_5 \times S^5$.

The expression for ϕ is more transparent in the vector notations. Let us think of V as the symmetric traceless tensor of $SO(6)$ (the upper latin indices) and the symmetric traceless tensor of $SO(2,4)$ (the lower greek indices). We parametrize the point of $AdS_5 \times S^5$ as a pair of vectors $(X, Y) \in \mathbf{R}^{2+4} \oplus \mathbf{R}^6$, $\|X\|^2 = \|Y\|^2 = 1$. We get:

$$\phi(X, Y) = X^{A_1} \dots X^{A_{n+1}} Y_{I_1} \dots Y_{I_{n+1}} V_{A_1 \dots A_{n+1}}^{I_1 \dots I_{n+1}} \quad (193)$$

9 Generalization

The construction of the vertex in the “airplane gauge” allows the following generalization.

9.1 General ansatz

Suppose that \mathcal{H}' is such that the second Casimir operator of \mathfrak{g} vanishes on \mathcal{H}' . Suppose that exists a vector $\Omega \in \mathcal{H}'$ which is:

1. annihilated by $\mathfrak{g}_0 = sp(2)_A \oplus sp(2)_S \subset \mathfrak{g}$
2. annihilated by \mathfrak{n}_-

The subalgebra $\mathfrak{g}_2 \subset \mathfrak{g}$ is the vector representation of $sp(2)_A$ plus the vector of $sp(2)_S$:

$$\mathfrak{g}_2 = \text{Vect}_A + \text{Vect}_S \quad (194)$$

The generators of \mathfrak{g}_2 will be denoted $t_{[\alpha\beta]}$ and $t^{[ab]}$. Let us consider:

$$\begin{aligned}
v(\lambda, \mu) = & ||\mu \cap \theta \cup \lambda \cap \theta|| \Omega + \\
& + (\theta \cap \lambda \cup \theta \cap \theta \cup \lambda \cap \theta)_{\omega-\text{less}}^{\alpha\beta} t_{[\alpha\beta]} \Omega + \\
& + (\theta \cup \lambda \cap \theta \cup \theta \cap \lambda \cup \theta)_{ab}^{\omega-\text{less}} t^{[ab]} \Omega + \\
& + (\theta \cap \mu \cup \theta \cap \theta \cup \mu \cap \theta)_{\omega-\text{less}}^{\alpha\beta} t_{[\alpha\beta]} \Omega + \\
& + (\theta \cup \mu \cap \theta \cup \theta \cap \mu \cup \theta)_{ab}^{\omega-\text{less}} t^{[ab]} \Omega + \\
& + [\text{terms of the order } \lambda\mu\theta^{\geq 4}, \lambda^2\theta^{\geq 6} \text{ and } \mu^2\theta^{\geq 6}]
\end{aligned} \tag{195}$$

This is the generalization of (177); comparison with (143) shows that in our explicit finite-dimensional construction Ω is the product of the Roiban-Siegel symplectic forms:

$$\Omega = (\omega^{\bullet\bullet})^{\otimes(n+1)} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} \tag{196}$$

9.2 Deriving the general ansatz in the airplane gauge

9.2.1 Constructing the BRST closed expression

Notice that Q_+ is symmetric with respect to the exchange $\lambda \leftrightarrow \mu$, and Q_- is antisymmetric. Therefore we have the E -symmetry (172) of the BRST complex. Observe that $E^2 = (-)^{\#\theta}$ and:

$$EQ_L = -iQ_RE \tag{197}$$

$$EQ_R = -iQ_LE \tag{198}$$

$$EQ_LQ_R = Q_LQ_RE \tag{199}$$

Observe that $||\mu \cap \theta \cup \lambda \cap \theta||$ is meta-odd and E -even. We get, as in Section 7.3:

$$\begin{aligned}
& Q_L ||\mu \cap \theta \cup \lambda \cap \theta|| \Omega = \\
& = Q_R \left((\theta \lambda \theta \theta \lambda \theta)^{\alpha\beta} T_{\alpha\beta} \Omega + (\theta \lambda \theta \theta \lambda \theta)_{ab} T^{ab} \Omega + Q_L A_{[\lambda\theta^5]} + X_{[\lambda^2\theta^6+\dots]} \right) + \\
& + Y_{[\mu\lambda^2\theta^5+\dots]}
\end{aligned} \tag{200}$$

where $X_{[\lambda^2\theta^6+\dots]}$ is such that:

$$Q_L \left((\theta \lambda \theta \theta \lambda \theta)^{\alpha\beta} T_{\alpha\beta} \Omega + (\theta \lambda \theta \theta \lambda \theta)_{ab} T^{ab} \Omega + X_{[\lambda^2\theta^6+\dots]} \right) = 0 \tag{201}$$

The obstacle to the existence of such $X_{[\lambda^2\theta^6+\dots]}$ is:

$$(\theta\lambda\theta\theta\lambda\theta)^{\alpha\beta}(\theta\lambda)^{\gamma\delta}T_{\alpha\beta}T_{\gamma\delta}\Omega + (\theta\lambda\theta\theta\lambda\theta)_{ab}(\theta\lambda)_{cd}T^{ab}T^{cd}\Omega \quad (202)$$

and therefore it is of the type $\lambda^3\theta^5$. It vanishes when the bosonic quadratic Casimir vanishes on Ω ; the bosonic quadratic Casimir vanishes on Ω because the full quadratic Casimir of \mathbf{g} vanishes on \mathcal{H}' , and Ω is annihilated by \mathbf{n}_- .

Notice that $Y_{[\mu\lambda^2\theta^5+\dots]}$ in Eq. (200) is Q_L -closed and therefore Q_L -exact:

$$Y_{[\mu\lambda^2\theta^5+\dots]} = Q_L Z_{[\mu\lambda\theta^6+\dots]} \quad (203)$$

Observe that $Q_R Q_L Z_{[\mu\lambda\theta^6+\dots]}$ is E -even. Therefore we get:

$$Q_L Q_R (Z - EZ) = 0 \quad (204)$$

This implies the existence of $U_{[\mu^2\theta^6+\dots]}$ and $V_{[\lambda^2\theta^6+\dots]}$ such that:

$$Q_R (Z - EZ) = Q_L U_{[\mu^2\theta^6+\dots]} \quad (205)$$

$$Q_L (Z - EZ) = Q_R V_{[\lambda^2\theta^6+\dots]} \quad (206)$$

Observe that $Q_L V = 0$ and therefore exists $W_{[\lambda\theta^7+\dots]}$ such that:

$$V = Q_L W_{[\lambda\theta^7+\dots]} \quad (207)$$

We get:

$$Q_L (Z - EZ + Q_R W) = 0 \quad (208)$$

Now Eq. (203) gives:

$$Y_{[\mu\lambda^2\theta^5+\dots]} = Q_L (EZ_{[\mu\lambda\theta^6+\dots]} - Q_R W_{[\lambda\theta^7+\dots]}) \quad (209)$$

Now we get:

$$\begin{aligned} Q_L (||\mu \cap \theta \cup \lambda \cap \theta||\Omega - EZ + Q_R W) &= \\ &= Q_R ((\theta\lambda\theta\theta\lambda\theta)^{\alpha\beta}T_{\alpha\beta}\Omega + (\theta\lambda\theta\theta\lambda\theta)_{ab}T^{ab}\Omega + Q_L A_{[\lambda\theta^5]} + X_{[\lambda^2\theta^6+\dots]}) \end{aligned} \quad (210)$$

and:

$$\begin{aligned} Q_R (||\mu \cap \theta \cup \lambda \cap \theta||\Omega - EZ + Q_R W) &= \\ &= Q_L ((\theta\mu\theta\theta\mu\theta)^{\alpha\beta}T_{\alpha\beta}\Omega + (\theta\mu\theta\theta\mu\theta)_{ab}T^{ab}\Omega - iQ_R(EA)_{[\mu\theta^5]} + (EX)_{[\mu^2\theta^6+\dots]}) \end{aligned} \quad (211)$$

The second equality was derived in the following way:

$$\begin{aligned}
Q_R (||\mu \cap \theta \cup \lambda \cap \theta||\Omega - EZ + Q_R W) &= iE Q_L (||\mu \cap \theta \cup \lambda \cap \theta||\Omega - Z) = \\
&= iE Q_R ((\theta \lambda \theta \theta \lambda \theta)^{\alpha\beta} T_{\alpha\beta} \Omega + (\theta \lambda \theta \theta \lambda \theta)_{ab} T^{ab} \Omega + Q_L A_{[\lambda\theta^5]} + X_{[\lambda^2\theta^6+\dots]}) = \\
&= Q_L ((\theta \mu \theta \theta \mu \theta)^{\alpha\beta} T_{\alpha\beta} \Omega + (\theta \mu \theta \theta \mu \theta)_{ab} T^{ab} \Omega - iQ_R(EA)_{[\mu\theta^5]} + (EX)_{[\mu^2\theta^6+\dots]})
\end{aligned}$$

Let us therefore consider the following expression:

$$\begin{aligned}
&||\mu \cap \theta \cup \lambda \cap \theta||\Omega - EZ + Q_R W - \\
&- ((\theta \lambda \theta \theta \lambda \theta)^{\alpha\beta} T_{\alpha\beta} \Omega + (\theta \lambda \theta \theta \lambda \theta)_{ab} T^{ab} \Omega + X_{[\lambda^2\theta^6+\dots]} + Q_L A_{[\lambda\theta^5]}) - \quad (212) \\
&- ((\theta \mu \theta \theta \mu \theta)^{\alpha\beta} T_{\alpha\beta} \Omega + (\theta \mu \theta \theta \mu \theta)_{ab} T^{ab} \Omega + (EX)_{[\mu^2\theta^6+\dots]} - iQ_R(EA)_{[\mu\theta^5]})
\end{aligned}$$

Eqs. (210) and (211) imply that this expression is Q -closed.

Also observe that the terms $Q_L A_{[\lambda\theta^5]}$ and $iQ_R(EA)_{[\mu\theta^5]}$ are gauge equivalent to expressions of the form $[\lambda\mu\theta^4 + \dots]$.

9.2.2 Showing that the constructed expression is not BRST exact

Let us prove that it is not Q -exact. Assume that we have found ϕ such that $Q\phi = v$. Then in particular:

$$\begin{aligned}
Q_L \phi_\lambda &= (\theta \cap \lambda \cup \theta \cap \theta \cup \lambda \cap \theta)^{\alpha\beta}_{\omega\text{-less}} t_{[\alpha\beta]} \Omega + \\
&+ (\theta \cup \lambda \cap \theta \cup \theta \cap \lambda \cup \theta)^{\omega\text{-less}}_{ab} t^{[ab]} \Omega + \dots \quad (213)
\end{aligned}$$

Such ϕ_λ should start with the leading term $\lambda\theta$. But Q_{R+} on the leading term will then give $\lambda\mu$. This means that ϕ in fact does not gauge away v , but actually rather brings it to the “rocket” gauge (*i.e.* removes the $\lambda\lambda$ and $\mu\mu$ -parts at the expense of introducing the θ -less term in the $\lambda\mu$ -part).

Another proof can be given by observing the nontrivial dilaton profile, using the methods of Section 8.2.

9.3 Supergravity meaning of Ω

This generalization of our construction described in Section 9 also works for infinite-dimensional representations. But for infinite-dimensional representations the construction of Ω is less transparent than the explicit formula (196).

There is a candidate Ω for \mathcal{H} being the space of all linearized supergravity solutions. Let us think of \mathcal{H}' as the space of all gauge-invariant SUGRA operators at the fixed point of $AdS_5 \times S^5$. We can restrict ourselves to evaluating them on a particular subspace. The supersymmetry transformations of the supergravity fields can be found *e.g.* in [12]. In particular, the transformation laws for the dilaton-axion field V_-^α is:

$$\delta V_-^\alpha = \kappa V_+^\alpha \bar{\eta} \lambda^* \quad (214)$$

Take a gauge invariant combination, for example V_-^1/V_-^2 . Then (214) implies that this combination is invariant under the (complexified) supersymmetry transformations which have $\bar{\eta} = 0$ (and the only nonzero parameter is $\bar{\eta}^*$).

This means that this operator is annihilated by \mathbf{n}_- . Therefore we can take Ω in the following form:

$$\Omega : \mathcal{H} \rightarrow \mathbb{C}$$

$$\Omega \left(\begin{array}{c} \text{SUGRA} \\ \text{solution} \end{array} \right) = \left[\begin{array}{l} \text{fluctuation of } V_-^1/V_-^2 \\ \text{evaluated on this solution} \\ \text{at the marked point of } AdS_5 \times S^5 \end{array} \right] \quad (215)$$

Here “fluctuation” of the field means the difference with the value in the undeformed $AdS_5 \times S^5$. This is, essentially, a complex linear combination of the fluctuation of dilaton plus axion:

$$\Omega = \delta\phi + i\psi \quad (216)$$

Then we can construct the vertex using (195).

10 Subspaces and factorspaces

10.1 Equivalence relation?

As we explained in Section 2.4 our space \mathcal{H} is not irreducible. The kernel of the map $\text{ev} : \mathcal{H} \rightarrow \mathcal{T}$ is an invariant subspace, and there is no complementary subspace. Because the space of deformations is not a unitary representation, there is no obvious reason why it should be irreducible. It is natural to ask the following question:

- Is it possible to define the vertex on the irreducible representation $\mathcal{T} = \mathcal{H}/\ker(\text{ev})$ (the Young diagramm representation) rather than \mathcal{H} ?

This would be possible iff the covariant vertex $v(\lambda, \mu)$ which we constructed were Q -exact up to $(\ker(\text{ev}))^\perp$:

$$v \stackrel{?}{\in} \text{Im}(Q) + (\ker(\text{ev}))^\perp \quad (217)$$

— see Eq. (28). If this conjecture was true, then this would imply that the “dressed” vertex $V[\Psi](g, \lambda)$ given by (3) is Q -exact when Ψ is in $\ker(\text{ev})$. This would imply that the space of linearized deformations is really \mathcal{T} of (23) rather than \mathcal{H} of (11).

However the hypothesis (217) is not true. Indeed, let us consider the “airplane” gauge of Eq. (177). We can remove the $\lambda\lambda$ and $\mu\mu$ parts (“the wings”) mod $(\ker(\text{ev}))^\perp$. Indeed, let us look at the structure of the $\lambda\lambda$ wing. It consists of the terms like this one:

$$\bullet\theta \cap \lambda \cup \theta \cap \theta \cup \lambda \cap \theta^\bullet \otimes (\omega^{\bullet\bullet})^{\otimes n} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} \quad (218)$$

This is Q_{L+} -equivalent to

$$\bullet\theta \cap \lambda \cup \theta_{\bullet\bullet} \cup \lambda \cap \theta^\bullet \otimes (\omega^{\bullet\bullet})^{\otimes(n+1)} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} \quad (219)$$

Here all the θ ’s enter with one uncontracted index, therefore this is in $(\ker(\text{ev}))^\perp$. Therefore:

$$v = \frac{5}{8n+8} \|\mu \cap \theta \cup \lambda \cap \theta\| (\omega^{\bullet\bullet})^{\otimes(n+1)} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} \text{ mod } (\ker(\text{ev}))^\perp \quad (220)$$

Notice that this expression is BRST closed modulo $(\ker(\text{ev}))^\perp$, but not BRST exact modulo $(\ker(\text{ev}))^\perp$. Therefore the hypothesis (217) is false.

Proof that v is not BRST exact modulo $(\ker(\text{ev}))^\perp$ By the symmetries the only candidate for $Q^{-1}v$ would be:

$$\|\mu \cap \theta\| (\omega^{\bullet\bullet})^{\otimes(n+1)} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} - \|\lambda \cap \theta\| (\omega^{\bullet\bullet})^{\otimes(n+1)} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} \quad (221)$$

(All expressions in this paragraph are mod $(\ker(\text{ev}))^\perp$.) Consider the $\lambda\lambda$ part of Q of (221):

$$\|\lambda \cap \theta\| (\{\lambda \cap \theta\}^{\bullet\bullet} \omega_{\bullet\bullet} - \{\lambda \cup \theta\}_{\bullet\bullet} \omega^{\bullet\bullet}) \otimes (\omega^{\bullet\bullet} \otimes \omega_{\bullet\bullet})^{\otimes n} \quad (222)$$

But this is not Q_{L+} -exact. Indeed, by symmetries, the only candidates are

$$\begin{aligned}
& ||\lambda \cap \theta \cup \theta \cap \theta|| (\omega^{\bullet\bullet} \otimes \omega_{\bullet\bullet})^{\otimes(n+1)} \\
& (\theta^\bullet \cap \lambda \cup \theta \cap \theta^\bullet - \theta^\bullet \cap \theta \cup \lambda \cap \theta^\bullet) (\omega^{\bullet\bullet})^{\otimes n} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} \\
& (\lambda^\bullet \cap \theta \cup \theta \cap \theta^\bullet - \theta^\bullet \cap \theta \cup \theta \cap \lambda^\bullet) (\omega^{\bullet\bullet})^{\otimes n} \otimes (\omega_{\bullet\bullet})^{\otimes(n+1)} \\
& (\theta_\bullet \cup \lambda \cap \theta \cup \theta_\bullet - \theta_\bullet \cup \theta \cap \lambda \cup \theta_\bullet) (\omega^{\bullet\bullet})^{\otimes(n+1)} \otimes (\omega_{\bullet\bullet})^{\otimes n} \\
& (\lambda_\bullet \cup \theta \cap \theta \cup \theta_\bullet - \theta_\bullet \cup \theta \cap \theta \cup \lambda_\bullet) (\omega^{\bullet\bullet})^{\otimes(n+1)} \otimes (\omega_{\bullet\bullet})^{\otimes n} \quad (223)
\end{aligned}$$

but they do not give the right expression when acted on by Q_{L+} (in fact most of them are Q_{L+} -exact).

This proof does not work for the generalized construction described in Section 9, because we do not know how to generalize the step leading from Eq. (218) to Eq. (219).

10.2 Vertex for irreducible representations

We get the short exact sequence:

$$0 \rightarrow \ker(\text{ev}) \rightarrow \mathcal{H} \rightarrow \mathcal{T} \rightarrow 0 \quad (224)$$

Restricting our vertex on $\ker(\text{ev})$ we get a nontrivial element of $H^2(Q, \ker(\text{ev}))$. If $\ker(\text{ev})$ contains an invariant subspace, then we can repeat the process:

The reduction process. Generally speaking, suppose that we have an exact sequence of representations:

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \quad (225)$$

Suppose that we have constructed a universal covariant vertex transforming in B . Then there are 2 possibilities:

1. everything in A is exact (an “equivalence relation” like in Section 10.1),
or
2. there is no such equivalence relation, the restriction of the universal vertex to the states in A is nontrivial in cohomology

In the first case we get the vertex in $C = B/A$, and in the second case we get the vertex in A .

Eventually, repeating the process, we obtain a universal vertex for some irreducible representation. The realization of this process requires the detailed study of the structure of the Kac module along the lines of [13, 14, 15, 16, 17] and references therein.

11 Open questions

1. Understand the field theory side.
2. We have constructed the unintegrated vertex. It would be interesting to carry out the descent procedure and construct the integrated vertex, as was done in [8] for the β -deformation vertex.
3. Measure the supergravity fields corresponding to $v(\lambda, \mu)$ for the states Ψ more general than those studied in Section 8.
4. The generalization to infinite-dimensional representations described in Section 9 requires further study.
5. When $n = 0$, what is the relation between the vertex constructed in this paper and the vertex of [8]? (See Appendix B.)
6. Generally speaking, it would be interesting to study the vertex operators even in *flat space* (in pure spinor, or Green-Schwarz, or NSR formalism), corresponding to the linearized SUGRA solutions *polynomial* in the coordinates. This would be the flat space limit of our construction, as discussed in Section 8.2.

Acknowledgments

We would like to thank N. Berkovits and V. Pershin for useful discussions. This work was supported in part by the Ministry of Education and Science of the Russian Federation under contract 14.740.11.0347, and in part by the RFFI grant 10-02-01315.

A Gamma-matrix expressions

A.1 Correcting $\overset{0}{\Psi}_{\text{as}}$

Consider:

$$\overset{0}{\Psi}_{\text{as}} = (\mu \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma_m \theta) (\lambda \Gamma^m \theta) \quad (226)$$

$$\overset{0}{\Xi}_{\text{as}} = \frac{1}{2} (\theta \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma_m \theta) (\mu \Gamma^m \theta) \quad (227)$$

Then we have:

$$\begin{aligned} \overset{0}{\Psi}_{\text{as}} + Q_{L+} \overset{0}{\Xi}_{\text{as}} &= (\mu \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma^m \theta) (\lambda \Gamma_m \theta) + \frac{1}{2} (\lambda \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma^m \theta) (\mu \Gamma_m \theta) - \\ &\quad - \frac{1}{2} (\theta \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma^m \lambda) (\mu \Gamma_m \theta) + \frac{1}{2} (\theta \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma^m \theta) (\mu \Gamma_m \lambda) \end{aligned} \quad (228)$$

Notice that:

$$\begin{aligned} &\frac{1}{2} (\lambda \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma^m \theta) (\mu \Gamma_m \theta) - \frac{1}{2} (\theta \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma^m \lambda) (\mu \Gamma_m \theta) = \\ &= (\lambda \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma^m \theta) (\mu \Gamma_m \theta) - (\theta \Gamma_{\mathbf{a}} \lambda) (\mu \Gamma_{\mathbf{s}} \theta) - (\theta \Gamma_{\mathbf{a}} \mu) (\lambda \Gamma_{\mathbf{s}} \theta) \end{aligned} \quad (229)$$

Therefore:

$$\begin{aligned} \overset{0}{\Psi}_{\text{as}} + Q_L \overset{0}{\Xi}_{\text{as}} &= (\mu \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma^m \theta) (\lambda \Gamma_m \theta) - (\theta \Gamma_{\mathbf{a}} \lambda) (\mu \Gamma_{\mathbf{s}} \theta) + \frac{1}{4} (\theta \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma^m \theta) (\mu \Gamma_m \lambda) + \\ &\quad + (\lambda \leftrightarrow \mu) \end{aligned} \quad (230)$$

We conclude that $\overset{0}{\Psi}_{\text{as}} + Q_L \overset{0}{\Xi}_{\text{as}}$ is symmetric with respect to $(\lambda \leftrightarrow \mu)$. We can also rewrite it as follows:

$$\overset{0}{\Psi}_{\text{as}} + Q_L \overset{0}{\Xi}_{\text{as}} = 6(\theta \Gamma_{[\mathbf{a}} \lambda)(\mu \Gamma_{\mathbf{s}]} \theta) + \frac{3}{2} (\theta \Gamma_{\mathbf{a}} \Gamma_{\mathbf{s}} \Gamma_m \theta) (\lambda \Gamma^m \mu) \quad (231)$$

A.2 With $\theta = \theta_L + \theta_R$

Now let us investigate the trace part:

$$(\lambda \Gamma^m \theta) (\theta \Gamma_m \mu) \quad (232)$$

We substitute $\theta = \theta_L + \theta_R$. Notice that the $\theta_L \theta_L$ part is Q -exact:

$$(\lambda \Gamma^m \theta_L) (\theta_L \Gamma_m \mu) = Q((\lambda \Gamma^m \theta_L) (\theta_L \Gamma_m \theta_R)) \quad (233)$$

and similarly is the $\theta_R\theta_R$ term. The $\theta_R\theta_L$ part is:

$$(\lambda\Gamma^m\theta_L)(\theta_R\Gamma_m\mu) + (\lambda\Gamma^m\theta_R)(\theta_L\Gamma_m\mu) \quad (234)$$

Observe:

$$\begin{aligned} (\lambda\Gamma^m\theta_R)(\theta_L\Gamma_m\mu) &= Q((\lambda\Gamma^m\theta_R)(\theta_L\Gamma_m\theta_R)) - (\lambda\Gamma^m\mu)(\theta_L\Gamma_m\theta_R) = \\ &= Q((\lambda\Gamma^m\theta_R)(\theta_L\Gamma_m\theta_R)) + (\mu\Gamma^m\theta_L)(\lambda\Gamma_m\theta_R) + (\lambda\Gamma^m\theta_L)(\mu\Gamma_m\theta_R) \end{aligned} \quad (235)$$

which implies that:

$$(\lambda\Gamma^m\theta_R)(\theta_L\Gamma_m\mu) = \frac{1}{2}(\lambda\Gamma^m\theta_L)(\mu\Gamma_m\theta_R) + Q(\dots) \quad (236)$$

Therefore:

$$(\lambda\Gamma^m\theta)(\theta\Gamma_m\mu) = \frac{3}{2}(\lambda\Gamma^m\theta_L)(\mu\Gamma_m\theta_R) + Q(\dots) \quad (237)$$

B Beta deformation

Here we will rewrite the β -deformation vertex of [8] using our current notations.

We start with the $\lambda\lambda$ part:

$$(V_{LL})_{ab}^{\alpha\beta} = \theta^{[\alpha} \cap \lambda \cup \theta_{[a} \theta_{b]} \cup \lambda \cap \theta^{\beta]} \quad (238)$$

Note that this is the complete expression, annihilated by Q_L , there is no need to add the terms of the higher order in θ .

Now let us proceed to the $\lambda\mu$ part. We act on V_{LL} by Q_R and see if it is cancelled by Q_L of something. Notice that $Q_R V_{LL} = 0$, and therefore it is enough to calculate $Q_{R+} V_{LL}$:

$$\begin{aligned} (Q_R V_{LL})_{ab}^{\alpha\beta} &= (Q_{R+} V_{LL})_{ab}^{\alpha\beta} = 2 \mu^{[\alpha} \cap \lambda \cup \theta_{[a} \theta_{b]} \cup \lambda \cap \theta^{\beta]} - \\ &\quad - 2 \theta^{[\alpha} \cap \lambda \cup \mu_{[a} \theta_{b]} \cup \lambda \cap \theta^{\beta]} \end{aligned} \quad (239)$$

We observe:

$$(Q_R V_{LL})_{ab}^{\alpha\beta} = Q_L \left(2 \mu_{[a}^{[\alpha} \theta_{b]} \cup \lambda \cap \theta^{\beta]} \right) \quad (240)$$

On the other hand:

$$\begin{aligned} Q_R \left(2\mu_{[a}^{[\alpha} \theta_{b]} \cup \lambda \cap \theta^{\beta]} \right) = & 2\mu_{[a}^{[\alpha} \mu_{b]} \cup \lambda \cap \theta^{\beta]} - \\ & - 2\mu_{[a}^{[\alpha} \theta_{b]} \cup \lambda \cap \mu^{\beta]} \end{aligned} \quad (241)$$

Finally:

$$2\mu_{[a}^{[\alpha} \mu_{b]} \cup \lambda \cap \theta^{\beta]} - 2\mu_{[a}^{[\alpha} \theta_{b]} \cup \lambda \cap \mu^{\beta]} = Q_L \left(\mu_{[a}^{[\alpha} \mu_{b]}^{\beta]} \right) \quad (242)$$

Therefore the following expression is Q -closed:

$$\mu_{[a}^{[\alpha} \mu_{b]}^{\beta]} - 2\mu_{[a}^{[\alpha} \theta_{b]} \cup \lambda \cap \theta^{\beta]} + \theta^{[\alpha} \cap \lambda \cup \theta_{[a} \theta_{b]} \cup \lambda \cap \theta^{\beta]} \quad (243)$$

There is also a symmetric version:

$$\begin{aligned} & \mu_{[a}^{[\alpha} \mu_{b]}^{\beta]} + \lambda_{[a}^{[\alpha} \lambda_{b]}^{\beta]} + 2\lambda_{[a}^{[\alpha} \theta_{b]} \cup \mu \cap \theta^{\beta]} - 2\mu_{[a}^{[\alpha} \theta_{b]} \cup \lambda \cap \theta^{\beta]} + \\ & + \theta^{[\alpha} \cap \lambda \cup \theta_{[a} \theta_{b]} \cup \lambda \cap \theta^{\beta]} + \theta^{[\alpha} \cap \mu \cup \theta_{[a} \theta_{b]} \cup \mu \cap \theta^{\beta]} \end{aligned} \quad (244)$$

This expression is BRST-equivalent to the vertex operator of the β -deformation studied in [8]. Indeed, the vertex operator of [8] is:

$$\begin{aligned} V_{\text{beta}} &= (\lambda_3 - \lambda_1) \wedge (\lambda_3 - \lambda_1) = \\ &= (\lambda_{3+} - \lambda_{1+} + \lambda_{3-} - \lambda_{1-}) \wedge (\lambda_{3+} - \lambda_{1+} + \lambda_{3-} - \lambda_{1-}) \end{aligned} \quad (245)$$

On the other hand, notice that the following expressions are both Q -exact:

$$X = (\lambda_{1+} - \lambda_{1-} + \lambda_{3+} - \lambda_{3-}) \wedge (\lambda_{1+} + \lambda_{1-} - \lambda_{3+} - \lambda_{3-}) \quad (246)$$

and

$$Y = (\lambda_{1+} - \lambda_{1-} + \lambda_{3+} - \lambda_{3-}) \wedge (\lambda_{1+} - \lambda_{1-} + \lambda_{3+} - \lambda_{3-}) \quad (247)$$

We observe that $V_{\text{beta}} + 2X + Y$ equals 4 times (243). This shows that indeed (243) is identified with the vertex of [8]. However (243) does not coincide with the particular case of our general construction (195) specified to $n = 0$. It appears that for $n = 0$ we have two different vertices, namely (195) and (243). The relation between the two remains to be investigated. We suspect that they both correspond to the β -deformation, but in different gauges.

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